## 3-D Geometry (overview)

Vectors and points are represented by a *triplet* on numbers, e.g. (2,1,-4).

Length (magnitude): $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$.

Unit vectors (lenght 1).

$\mathbf{e}_1$ ($\mathbf{e}_2$, $\mathbf{e}_3$) is the unit vector of the $+x$ ($+y$, $+z$) direction, respectively.

Zero vector.

Scalar multiplication, e.g. $3 \cdot (2, -1, 4) = (6, -3, 12)$.

Addition (component-wise).

**Dot** (inner) [scalar] **product** of two vectors is *defined* by

$$\mathbf{a} \bullet \mathbf{b} \equiv |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\gamma$$

and is the length of the *projection* of $\mathbf{a}$ into the direction of $\mathbf{b}$, multiplied by $|\mathbf{b}|$ (or reverse).

Can be *computed* by

$$\mathbf{a} \bullet \mathbf{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$$

Proof based on distributive law:

$$(\mathbf{a} + \mathbf{b}) \bullet \mathbf{c} \equiv \mathbf{a} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c}$$

(clear geometrically).

**Cross** (outer) [vector] **product** $\mathbf{a} \times \mathbf{b}$ is defined as a *vector* whose length is $|\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin\gamma$ (area of parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as sides), whose direction is perpendicular (*orthogonal*) to each $\mathbf{a}$ and $\mathbf{b}$, and whose orientation is such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ follow the right-handed pattern. [Note it's *anti*-commutative, i.e. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

Another way: Project $\mathbf{a}$ into the plane perpendicular to $\mathbf{b}$, rotate this projection by +90$^o$ (counterclockwise) and multiply the resulting vector by $|\mathbf{b}|$.

Also note that this product is *not* associative: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Can be *computed* by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\mathbf{e}_3 \equiv$$

$$(a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$$

[e.g. $(1, 3, -2) \times (4, -2, 1) = (-1, -9, -14)$].

Proof based on distributive law:

$$(\mathbf{a}+\mathbf{b}) \times \mathbf{c} \equiv \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

Another way of expressing the $k^{th}$ component of $(\mathbf{a} \times \mathbf{b})$ is:

$$(\mathbf{a} \times \mathbf{b})_k = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \epsilon_{ijk}$$

(for $k = 1, 2, 3$), where $\epsilon_{ijk}$ [called a *fully antisymmetric* tensor] changes sign when any two indices are interchanged ($\Rightarrow \epsilon = 0$ unless $i$, $j$, $k$ distinct) and $\epsilon_{123} = 1$ (this defines the rest).

One can show that

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{j\ell}\delta_{im}$$

(where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$; this is Kronecker's delta).

Based on this result, one can prove several useful formulas such as, for example:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{b} \bullet \mathbf{c})\mathbf{a}$$

Proof:

$$\sum_{i,j,k,\ell} \epsilon_{ijk} a_i b_j \epsilon_{k\ell m} c_\ell = \sum_{i,j,\ell} (\delta_{i\ell}\delta_{jm} - \delta_{j\ell}\delta_{im}) a_i b_j c_\ell$$
$$= \sum_{\ell} (a_\ell b_m c_\ell - a_m b_\ell c_\ell)$$

and

$(\mathbf{a}\times\mathbf{b})\bullet(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\bullet\mathbf{c})(\mathbf{b}\bullet\mathbf{d}) - (\mathbf{a}\bullet\mathbf{d})(\mathbf{b}\bullet\mathbf{c})$ having a similar proof.

**Triple product** of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is, by definition, equal to $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$ and can be computed by $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

It represents the volume of the parallelepiped with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ being three of its sides (further multiplied by $-1$ if the three vectors constitute a *left-handed* set).

This implies that $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \bullet (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b}) = -\mathbf{b} \bullet (\mathbf{a} \times \mathbf{c}) = -\mathbf{c} \bullet (\mathbf{b} \times \mathbf{a}) = \mathbf{a} \bullet (\mathbf{c} \times \mathbf{b})$ [its value does not change under *cyclic* permutation of the three vectors].

A useful application of the triple product is the following test: $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are in the same plane (*co-planar*) iff $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = 0$.

Another one is to compute the volume of an (arbitrary) tetrahedron. Note that if you use the three vectors as sides of the tetrahedron (instead of parallelepiped), its base will be half of the parallelepiped's, and its volume will thus be $\frac{\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})}{6}$.

Rotation is facilitated by an *orthogonal* matrix $\mathbb{R}$ whose determinant is $+1$, thus $\mathbb{R}\mathbf{a}$.

## Straight Lines and Planes

There are two ways of defining a straight line:

(i) **parametric representation**, i.e. $\mathbf{a} + \mathbf{b} \cdot t$ where $\mathbf{a}$ is an arbitrary point on the straight line and $\mathbf{b}$ is a vector along its direction, and $t$ (the actual *parameter*) is a scalar allowed to vary from $-\infty$ to $+\infty$

(ii) by **two linear equations**, e.g. $\begin{cases} 2x + 3y - 4z = 6 \\ x - 2y + z = -2 \end{cases}$

(effectively an intersection of two planes).

Neither description is unique (a headache when marking assignments).

Similarly, there are two ways of defining a plane:

(i) **parametric**, i.e. $\mathbf{a} + \mathbf{b} \cdot u + \mathbf{c} \cdot v$ where $\mathbf{a}$ is an arbitrary point in the plane, $\mathbf{b}$ and $\mathbf{c}$ are two nonparallel vectors within the plane, and $u$ and $v$ are scalar parameters varying over all possible real values

(ii) by a **single linear equation**, e.g. $2x + 3y - 4z = 6$. Note that $(2, 3, -4)$ is a vector *perpendicular* to the plane, its so called *normal* – to prove it, substitute two distinct points into the equation and subtract, getting the dot product of the connecting vector and $(2, 3, -4)$, always equal to zero.

Again, neither description is unique.

EXAMPLES:

1. Convert $\left\{ \begin{array}{c} 3x + 7y - 4z = 5 \\ 2x - 3y + z = -4 \end{array} \right.$ to its

parametric representation.

Solution: The cross product of the two normals must point along the straight line, giving us
$\mathbf{b} = (3, 7, -4) \times (2, -3, 1) = (-5, -11, -23)$
Solving the two equations with an arbitrary value of $z$ (say $= 0$) yields $\mathbf{a} = (-\frac{13}{23}, \frac{22}{23}, 0)$.

Answer: $(-\frac{13}{23} - 5t, \frac{22}{23} - 11t, -23t)$.

2. Find a equation of an (infinite) cylindrical surface with $(3 - 2t, 1 + 3t, -4t)$ as its axis, and with the radius of 5.

Solution: Let us first find an expression of the (shortest) distance from a point $\mathbf{r} \equiv (x, y, z)$ to a straight line $\mathbf{a} + \mathbf{b} \cdot t$ [bypassing minimization]. Visualize the vector $\mathbf{r} - \mathbf{a}$. We know that $|\mathbf{r} - \mathbf{a}|$ is its length, and that $(\mathbf{r} - \mathbf{a}) \bullet \frac{\mathbf{b}}{|\mathbf{b}|}$ is the length of its projection into the straight line. By Pythagoras, the *direct* distance is

$$\sqrt{|\mathbf{r}-\mathbf{a}|^2-\left[(\mathbf{r}-\mathbf{a})\bullet\frac{\mathbf{b}}{|\mathbf{b}|}\right]^2}=$$

$$\sqrt{(x-3)^2+(y-1)^2+z^2-\frac{(-2x+3y-4z+3)^2}{29}}$$

Making this equal to $5$ yields the desired equation (square it to simplify).

Answer: $(x-3)^2+(y-1)^2+z^2-\frac{(-2x+3y-4z+3)^2}{29}=25$.

3. What is the (shortest) distance from $\mathbf{r}=(6,2,-4)$ to $3x-4y+z=7$ [bypass minimization].

Solution: $\mathbf{n}\bullet(\mathbf{r}-\mathbf{a})$, where $\mathbf{n}$ is the unit normal and $\mathbf{a}$ is an arbitrary point of the plane [found, in this case, by setting $x=y=0\Rightarrow(0,0,7)$].

Answer: $\frac{(3,-4,1)}{\sqrt{9+16+1}}\bullet(6,2,-11)=-\frac{1}{\sqrt{26}}$ [the minus sign establishes on which side of the plane we are].

4. Find the (shortest) distance between $\mathbf{a}_1+\mathbf{b}_1\cdot t$ and $\mathbf{a}_2+\mathbf{b}_2\cdot t$ [bypassing minimization, as always].

Solution: To find it, we have to move perpendicularly to both straight lines, i.e. along $\mathbf{b}_1 \times \mathbf{b}_2$. We also know that $\mathbf{a}_2 - \mathbf{a}_1$ is an arbitrary connection between the two lines. The projection of this vector into the direction of $\mathbf{b}_1 \times \mathbf{b}_2$ supplies (up to the sign) the answer: $(\mathbf{a}_2 - \mathbf{a}_1) \bullet \frac{\mathbf{b}_1 \times \mathbf{b}_2}{|\mathbf{b}_1 \times \mathbf{b}_2|}$ [visualize the situation by projecting the two straight lines into the blackboard so that they *look* parallel – always possible].

## Curves

are defined via their **parametric representation** $\mathbf{r}(t) \equiv [x(t), y(t), z(t)]$, where $x(t)$, $y(t)$ and $z(t)$ are arbitrary (continuous) functions of $t$ (the parameter, ranging over some interval of real numbers).

EXAMPLE: $\mathbf{r}(t) = [\cos(t), \sin(t), t]$ is a *helix* centered on the $z$-axis, whose radius (when projected into the $x$-$y$ plane) equals 1, with one full loop per $2\pi$ of vertical distance. The same $\mathbf{r}(t)$ can be also seen as a *motion* of a point-like particle, where $t$ represents time. Note that $[\cos(2t), \sin(2t), 2t]$ represents a different motion (the particle is moving twice as fast), but the *same* curve (i.e. parametrization of a curve is far from unique).

**Arc's length**

('arc' meaning a specific segment of the curve). The three-component (*vector*) distance travelled between time $t$ and $t + dt$ ($dt$ infinitesimal) is $\mathbf{r}(t+dt) - \mathbf{r}(t) \approx \mathbf{r}(t) + \dot{\mathbf{r}}(t)\,dt + .... - \mathbf{r}(t) = \dot{\mathbf{r}}(t)\,dt + ....$, where the dots stand for terms proportional to $dt^2$ and higher [these give zero contribution in the $dt \to 0$ limit], and $\dot{\mathbf{r}}(t)$ represents the *componentwise* differentiation with respect to $t$ (the particle's *velocity*). This converts to $|\dot{\mathbf{r}}(t)|\,dt + ...$ in terms of the actual *scalar* distance (length). Adding all these infinitesimal distances (from time $a$ to time $b$ – these should correspond to the arc's end points) results in

$$\int_a^b |\dot{\mathbf{r}}(t)|\,dt$$

which is the desired formula for the total length.

EXAMPLES:

1. Consider the helix of the previous example. The length of one of its complete loops (say from $t = 0$ to $t = 2\pi$) is thus $\int_0^{2\pi} |[-\sin(t), \cos(t), 1]|\,dt = \int_0^{2\pi} \sqrt{\sin(t)^2 + \cos(t)^2 + 1}\,dt = 2\pi\sqrt{2}$.

2. The intersect of $x^2 + y^2 = 9$ (a cylinder) and $3x - 4y + 7z = 2$ (a plane) is an ellipse. How long is it?

Solution: First we need to parametrize it, thus: $\mathbf{r}(t) = [3\cos(t), 3\sin(t), \frac{2-9\cos(t)+12\sin(t)}{7}]$ where $t \in [0, 2\pi)$.

Answer: $\int\limits_0^{2\pi} |\dot{\mathbf{r}}|\, dt = \int\limits_0^{2\pi} \sqrt{9 + \left(\frac{9\sin t + 12\cos t}{7}\right)^2}\, dt$
which is an integration we cannot carry out analytically (just to remind ourselves that this can frequently happen). Numerically (using Maple), this equals 21.062.

A **tangent** (straight) **line** to a curve, at a point $\mathbf{r}(t_0)$ [$t_0$ being a specific value of the parameter] passes through $\mathbf{r}(t_0)$, and has the direction of $\dot{\mathbf{r}}(t_0)$ [the velocity]. Its parametric representation will be thus

$$\mathbf{r}(t_0) + \dot{\mathbf{r}}(t_0) \cdot u$$

[where $u$ is the parameter now, just to differentiate].

EXAMPLE: Using the same helix, at $t = 0$ its tangent line is $[1, u, u]$.

When $\mathbf{r}(t)$ is seen as a motion of a particle, $\dot{\mathbf{r}}(t) \equiv \mathbf{v}(t)$ gives the particle's (instantaneous, 3-D) **velocity**. $|\dot{\mathbf{r}}(t)|$ then yields its (*scalar*) **speed** [the speedometer reading]. It is convenient to rewrite $\mathbf{v}(t)$ as

$|\dot{\mathbf{r}}(t)| \cdot \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} \equiv$

$$|\dot{\mathbf{r}}(t)| \cdot \mathbf{u}(t)$$

[a product of its speed and *unit* direction].

The corresponding (3-D) **acceleration** is simply $\mathbf{a}(t) \equiv \ddot{\mathbf{r}}(t)$ [a double $t$-derivative]. It is more meaningful to decompose it into its '*tangential*' [the one observed on the speedometer, pushing you back into your seat] and '*normal*' [observed even at constant speeds, pushing you sideways – perpendicular to the motion] components. This is achieved by the product rule: $\frac{d\mathbf{v}(t)}{dt} = \frac{d|\dot{\mathbf{r}}(t)|}{dt} \cdot \mathbf{u}(t) + |\dot{\mathbf{r}}(t)| \cdot \frac{d\mathbf{u}(t)}{dt}$ [tangential and normal, respectively].

$\frac{d|\dot{\mathbf{r}}(t)|}{dt}$ can be simplified to $\frac{d}{dt}\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} = \frac{1}{2} \cdot \frac{2\dot{x}\ddot{x}+2\dot{y}\ddot{y}+2\dot{z}\ddot{z}}{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2+\dot{z}(t)}} = \frac{\dot{\mathbf{r}}\bullet\ddot{\mathbf{r}}}{|\dot{\mathbf{r}}(t)|} =$

$$\mathbf{u} \bullet \ddot{\mathbf{r}}$$

The normal acceleration is then most easily computed from

$$\ddot{\mathbf{r}} - (\mathbf{u} \bullet \ddot{\mathbf{r}})\mathbf{u}$$

[full minus tangential]. In this form it is trivial to verify that the normal acceleration is perpendicular to $\mathbf{u}$.

EXAMPLE: For our helix at $t = 0$, the speed is $\sqrt{2}$, $\mathbf{u} = [0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ and $\ddot{\mathbf{r}} = [-1, 0, 0] \Rightarrow$ zero tangential acceleration and $[-1, 0, 0]$ normal acceleration. ■

When interested in the *geometric* properties of a curve only, it is convenient to make its parametrization *unique* by introducing a **special parameter** $s$ (instead of $t$) which measures the actual *length travelled* along the curve, i.e.

$$s(t) = \int_0^t |\dot{\mathbf{r}}(t)| \; dt$$

where $\mathbf{r}(t)$ is the old parametrization.

Unfortunately, to carry out the details of such a 'reparametrization' is normally too difficult [to eliminate $t$, we would have to solve the previous equation for $t$ – but we don't know how to solve general equations]. Yet, the idea of this new 'uniform' (in the sense of the corresponding motion) parameter $s$ is still quite helpful, when we realize that the previous equation is equivalent to

$$\frac{ds}{dt} = |\dot{\mathbf{r}}(t)|$$

This further implies that, even though we don't have an explicit formula for $s(t)$, we know how to *differentiate* with respect to $s$, as

$$\frac{d}{ds} \equiv \frac{\frac{d}{dt}}{\frac{ds}{dt}} \equiv \frac{\frac{d}{dt}}{|\dot{\mathbf{r}}(t)|}$$

Note that our old $\mathbf{u} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|}$ [the unit velocity direction] can thus be defined simply as $\frac{d\mathbf{r}}{ds} \equiv \mathbf{r}'$ [prime will imply $s$-differentiation].

Using this new parameter $s$, we now *define* a few interesting geometrical properties (describing a curve and its behavior in space); we will immediately 'translate' these into the $t$-'language', as we normally parametrize curves by $t$ and not $s$:

**Curvature**

Let us first compute $\frac{d\mathbf{u}}{ds} \equiv \mathbf{r}''$ which corresponds to the rate of change of the unit direction per (scalar) distance travelled. The result is a vector which is always perpendicular to $\mathbf{u}$, as we will show shortly.

*Curvature* $\kappa$ is the *magnitude* of this $\mathbf{r}''$, and corresponds, geometrically, to the reciprocal of the radius of a tangent circle to the curve at a point [a circle with the same $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ – 6 independent conditions].

The main thing now is to figure out is: how do we compute curvature when our curve has the usual $t$-parametrization? This is not too difficult, as $\dfrac{d\mathbf{u}(s)}{ds} = \dfrac{\frac{d\mathbf{u}(t)}{dt}}{|\dot{\mathbf{r}}(t)|} = \dfrac{\frac{d}{dt}\frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}}{|\dot{\mathbf{r}}|} = \dfrac{\ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^2} - \dfrac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|^3} \cdot (\mathbf{u} \bullet \ddot{\mathbf{r}})$ [since $\dfrac{d|\dot{\mathbf{r}}|}{dt} = \frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = (\mathbf{u} \bullet \ddot{\mathbf{r}})$] =

$$\frac{\ddot{\mathbf{r}}(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}}) - \dot{\mathbf{r}}(\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}})}{|\dot{\mathbf{r}}|^4}$$

[this is easily seen to be $\dot{\mathbf{r}}$ perpendicular, as claimed].

To get $\kappa$, we need the corresponding magnitude:

$$\sqrt{\frac{(\ddot{\mathbf{r}} \bullet \ddot{\mathbf{r}})(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}})^2 + (\dot{\mathbf{r}} \bullet \dot{\mathbf{r}})(\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}})^2 - 2(\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}})^2(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}})}{(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}})^4}} =$$

$$\sqrt{\frac{(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}})(\ddot{\mathbf{r}} \bullet \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}})^2}{(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}})^3}}$$

This is the final formula for computing curvature.

EXAMPLE: For the same old helix, $(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}}) = 2$, $(\ddot{\mathbf{r}} \bullet \ddot{\mathbf{r}}) = 1$, and $(\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}}) = 0 \Rightarrow \kappa = \sqrt{\frac{1}{2^2}} = \frac{1}{2}$ [the same for all points of the helix – that seems to make sense; the tangent circles all have a radius of $2$].

A few **related definitions:**

>From what we already know $\mathbf{r}'' = \kappa \cdot \mathbf{p}$ where $\mathbf{p}$ is a *unit* vector we will call principal normal, automatically orthogonal to $\mathbf{u}$ and pointing towards the tangent circle's center. Furthermore, $\mathbf{b} = \mathbf{u} \times \mathbf{p}$ must thus be yet another *unit* vector, orthogonal to both $\mathbf{u}$

and $\mathbf{p}$. It is called the binormal vector (perpendicular to the tangent circle's plane).

One can show that the *rate of change* of $\mathbf{b}$ (per unit distance travelled), namely $\mathbf{b}'$ is a vector in the direction of $\mathbf{p}$, i.e. $\mathbf{b}' = -\tau \cdot \mathbf{p}$, where $\tau$ defines the so called **torsion** ('twist') of the curve at the corresponding point [$\tau$ is thus either $+$ or $-$ of the corresponding magnitude, the extra minus sign is just a convention].

Note that knowing a curve's curvature and torsion, we can 'reconstruct' the curve (by solving the corresponding set of differential equations), but we will not go into that.

We now derive a formula for computing $\tau$ based on the usual $\mathbf{r}(t)$-parametrization.

First: $\mathbf{b}' = \mathbf{u}' \times \mathbf{p} + \mathbf{u} \times \mathbf{p}' = \mathbf{0} + \mathbf{u} \times \left(\frac{\mathbf{u}'}{\kappa}\right)'$ [since $\mathbf{u}' \equiv \kappa \mathbf{p}$].

Then: $\tau = -\mathbf{p} \bullet \mathbf{b}' = -\left(\frac{\mathbf{u}'}{\kappa}\right) \bullet \left[\mathbf{u} \times \left(\frac{\mathbf{u}'}{\kappa}\right)'\right] = -\dfrac{\mathbf{u}' \bullet (\mathbf{u} \times \mathbf{u}'')}{\kappa^2} = \dfrac{\mathbf{u} \bullet (\mathbf{u}' \times \mathbf{u}'')}{\kappa^2}$.

And finally: $\mathbf{u} = \mathbf{r}' = \dot{\mathbf{r}} \frac{dt}{ds}$, $\mathbf{u}' =$

$\mathbf{r}'' = \ddot{\mathbf{r}}(\frac{dt}{ds})^2 + \dot{\mathbf{r}}\frac{d^2t}{ds^2}$ and $\mathbf{u}'' = \mathbf{r}''' = \dddot{\mathbf{r}}\left(\frac{dt}{ds}\right)^3 + 3\ddot{\mathbf{r}}\frac{dt}{ds} \cdot \frac{d^2t}{ds^2} + \dot{\mathbf{r}}\frac{d^3t}{ds^3}$.

Putting it together [and realizing that, whenever identical vectors 'meet' in a triple product, the result is zero], we get $\tau = \frac{\dot{\mathbf{r}}\bullet(\ddot{\mathbf{r}}\times\dddot{\mathbf{r}})}{\kappa^2}\left(\frac{dt}{ds}\right)^6 =$ [since $\frac{dt}{ds} = \frac{1}{|\dot{\mathbf{r}}|}$]

$$\frac{\dot{\mathbf{r}} \bullet (\ddot{\mathbf{r}}\times\dddot{\mathbf{r}})}{(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}})(\ddot{\mathbf{r}} \bullet \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}})^2}$$

which is our final formula for computing torsion.

Both the original definition and the final formula clearly imply that a *planar* curve has a *zero* torsion (identically).

EXAMPLE: For the helix $\dddot{\mathbf{r}} = (\sin t, -\cos t, 0) \Rightarrow \tau = \frac{1}{2}$.

**FIELDS**

A scalar field is a (single-valued) function of $x$, $y$ and $z$, e.g. $f(x, y, z) = \frac{x(y+3)}{z}$.

A vector field is a vector-valued fund-tion of $x$, $y$ and $z$ (i.e. three functions, which are interpreted as three components of a vector, thus: $\mathbf{g}(x, y, z) \equiv [g_1(x, y, z), g_2(x, y, z), g_3(x, y, z)]$, e.g. $[xy, \frac{z-3}{x}, \frac{y(x-4)}{z^2}]$.

An operator is a 'prescription' which takes a field and modifies it (usually, by computing its derivatives, in which case it is called a *differential* operator) to return another field.

The most important cases of operators are:

### Gradient

which converts a scalar field $f(x, y, z)$ into the following *vector* field:

$$(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \equiv \nabla f(x, y, z)$$

The $\nabla$-*operator* is usually called 'del' (sometimes 'nabla'), and has three components, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$.

It yields the direction of the fastest increase in $f(x, y, z)$ when starting at $(x, y, z)$; its magnitude provides the corresponding rate

(per unit length). This can be seen by rewriting the generalized Taylor expansion of $f$ at $\mathbf{r}$, thus:
$f(\mathbf{r}+\mathbf{h}) = f(\mathbf{r})+\mathbf{h}\bullet\boldsymbol{\nabla} f(\mathbf{r})+$quadratic (in $\mathbf{h}$) and higher-ord
When $\mathbf{h}$ is a *unit* vector, $\mathbf{h}\bullet\boldsymbol{\nabla} f(\mathbf{r})$ provides a so called **directional derivative** of $f$, i.e. the rate of its increase in the $\mathbf{h}$-direction (obviously the largest when $\mathbf{h}$ and $\boldsymbol{\nabla} f$ are parallel).

An interesting **geometrical application** is this: $f(x, y, z) = c$ usually defines a surface (a 3-D 'contour' of $f$; a simple extension of the $f(x, y) = c$ idea). The gradient, evaluated at a point of such a surface, is obviously *normal* (perpendicular) to the surface at that point.

EXAMPLE: Find the normal direction to $z^2 = 4(x^2 + y^2)$ [a cone] at $(1, 0, 2)$.
Solution: $f \equiv 4(x^2+y^2)-z^2 = 0$ defines the surface. $\boldsymbol{\nabla} f = (8x, 8y, -2z)$, evaluated at $(1, 0, 2)$ yields $(8, 0, -4)$, which is the answer. One may like to convert it to a *unit* vector,

and spell out its orientation (either inward or outward).

**Application to Physics**: If $\mathbf{r}(t)$ represents a *motion* of a particle and $f(x, y, z)$ a temperature of the 3-D media in which the particle moves, $\dot{\mathbf{r}} \bullet \boldsymbol{\nabla} f[\mathbf{r}(t)]$ is the rate of change (per unit of $\mathrm{time}$) of temperature as the particle experiences it [nothing but a chain rule]. To convert this into a spacial (per unit *length*) rate, one would have to divide the previous expression by $|\dot{\mathbf{r}}|$.

## Divergence

converts a vector field $\mathbf{g}(\mathbf{r})$ to the following scalar field:

$$\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z} \equiv \boldsymbol{\nabla} \bullet \mathbf{g}(\mathbf{r})$$

Its significance (to Physics) lies in the following **interpretation**: If $\mathbf{g}$ represents some *flow* [the direction and rate of a motion of some continuous fluid in space; the rate being established by measuring mass/sec./cm.$^2$

through an infinitesimal area perpendicular to its direction], then the divergence tells us the rate of *mass loss* from an (infinitesimal) volume at each point, *per volume* [mass/sec./cm.$^3$]. This can be seen by surrounding the point by an (infinitesimal) cube, and figuring out the in/out flow through each of its sides [$h^2 g_1(x + \frac{h}{2}, y, z)$ is the outflow from one of them, etc.].

EXAMPLE: Find $\nabla \bullet (x^2, y^2, z^2)$. Answer: $2x + 2y + 2z$.

## Curl

(sometimes also called rotation), applied to a vector field $\mathbf{g}$, converts it to yet another vector field, thus:

$$[\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}, \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}, \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}] \equiv \nabla \times \mathbf{g}$$

If $\mathbf{g}$ represents a flow, Curl($\mathbf{g}$) can then be visualized by holding an imaginary paddle-wheel at each point to see how fast the wheel rotates (its axis at the fastest rotation yields the curl's direction, the torque establishes the corresponding magnitude).

EXAMPLE: $\mathsf{Curl}(x, yz, -x^2 - z^2) = (-y, 2x, 0)$.

One can easily prove the following **trivial identities**:

$$\mathsf{Curl}\left[\mathsf{Grad}\left(f\right)\right] \equiv \mathbf{0}$$
$$\mathsf{Div}\left[\mathsf{Curl}\left(\mathbf{g}\right)\right] = 0$$

There are also several nontrivial identities, for illustration we mention one only:

$\mathsf{Div}(\mathbf{g}_1 \times \mathbf{g}_2) = \mathbf{g}_2 \bullet \mathsf{Curl}(\mathbf{g}_1) - \mathbf{g}_1 \bullet \mathsf{Curl}(\mathbf{g}_2)$

Divergence and gradient are frequently applied, consecutively, to a scalar field $f$, to create a new scalar field $\mathsf{Div}[\mathsf{Grad}(f)] \equiv \triangle f =$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

(where $\triangle$ is the so called **Laplace operator**). It measures how much the value of $f$ (at each point) deviates from its average over some infinitesimal surface [visualize a cube] centered at the point, per the surface's area (the exact answer is obtained in the limit, as the size of the cube approaches zero).

## LINE INTEGRALS

are of two types:

### Scalar (Type I) Integrals

where we are given a (scalar) function $f(x, y, z)$ and a curve $\mathbf{r}(t)$, and need to integrate $f$ over an arc of the curve (which now assumes the rôle of the $x$-axis). All it takes is to add the areas of the individual 'rectangles' of base $|\dot{\mathbf{r}}|\,dt$ and 'height' [which, unfortunately, has to be pictured in an extra $4^{th}$ dimension] $f[\mathbf{r}(t)]$, ending up with

$$\int_a^b f[\mathbf{r}(t)] \cdot |\dot{\mathbf{r}}(t)|\,dt$$

which is just an ordinary (scalar) integral of a single variable $t$. Note that the result is *independent* of the actual curve parametrization.

This kind of integration can used for *(spacial)* **averaging** of the $f$-values (over a segment of a curve). All we have to do is to divide the above integral by the arc's length $\int_a^b |\dot{\mathbf{r}}(t)|\,dt$:

$$\overline{f}_{sp} = \frac{\int_a^b f[\mathbf{r}(t)] \cdot |\dot{\mathbf{r}}(t)|\,dt}{\int_a^b |\dot{\mathbf{r}}(t)|\,dt}$$

To average in *time* (taking $\mathbf{r}(t)$ to be a motion of a particle) one would do

$$\overline{f}_{tm} = \frac{\int_a^b f[\mathbf{r}(t)]\,dt}{b-a}$$

instead.

The **symbolic notation** for this integral is

$$\int_C f(\mathbf{r})\,ds$$

$s$ being the special unique parameter which corresponds to the 'distance travelled', and $C$ stands for a specific segment of a curve. To evaluate this integral, we normally use a convenient (arbitrary) parametrization of the curve (the result must be the same), and carry out the integration in terms of $t$.

Two other possible **applications** are:

1. **Center of mass** of a wire-like object of uniform mass density:
$$\left[\frac{\int\limits_{\mathcal{C}} x\,ds}{\int\limits_{\mathcal{C}} ds}, \frac{\int\limits_{\mathcal{C}} y\,ds}{\int\limits_{\mathcal{C}} ds}, \frac{\int\limits_{\mathcal{C}} z\,ds}{\int\limits_{\mathcal{C}} ds}\right]$$
[the denominator is the total length $L$].
2. **Moment of inertia** of any such an object:
$$\frac{M}{L}\int\limits_{\mathcal{C}} d^2 \cdot ds$$
where $d(x, y, z)$ is distance from the axis of rotation. [Angular acceleration is torque divided by moment of inertia].

EXAMPLES:

- Evaluate $\int\limits_{C}(x^2 + y^2 + z^2)^2\,ds$ where $C \equiv (\cos t, \sin t, 3t)$ with $t \in (0, 2\pi)$ [one loop of a helix].

Solution:

$$\int_0^{2\pi} (\cos^2 t+\sin^2 t+9t^2)^2 \sqrt{(-\sin t)^2 + (\cos t)^2 + 9}\, dt$$

$$= \sqrt{10} \int_0^{2\pi} (1 + 18t^2 + 81t^4)\, dt$$

$$= \sqrt{10} \left[ t + 6t^3 + \frac{81}{5} t^5 \right]_0^{2\pi} = 5.0639 \times 10^5$$

- Find the center of mass of a half circle (the circumference only) of radius $a$.

  Solution: $\mathbf{r}(t) = [a \cos t, a \sin t, 0] \Rightarrow$
  $\int_{\mathcal{C}} y\, ds = a^2 \int_0^{\pi} \sin t\, dt = 2a^2.$

  Answer: The center of mass is at $[0, \frac{2a^2}{\pi a}, 0] = [0, 0.63662a, 0]$.

- Find the moment of inertia of a circle (circumference) of mass $M$ and radius $a$ with respect to an axis passing through its center and two of its points.

Solution: Using $\mathbf{r}(t) = [a\cos t, a\sin t, 0]$ and $y$ as the axis, we get $\int\limits_{\mathcal{C}} (x^2 + z^2)\, ds =$

$$a^3 \int\limits_0^{2\pi} \cos^2 t\, dt = \pi a^3.$$

Answer: $\frac{M}{2\pi a} \cdot \pi a^3 = \frac{Ma^2}{2}$.

## Vector (Type II) Integrals

Here, we are given a vector function $\mathbf{g}(x, y, z)$ [i.e. effectively three functions $g_1$, $g_2$ and $g_3$] which represents a *force* on a point particle at $(x, y, z)$, and a curve $\mathbf{r}(t)$ which represents the particle's 'motion'. We know (from Physics) that, when the particle is moved by an infinitesimal amount $d\mathbf{r}$, the energy it extracts from the field equals $\mathbf{g} \bullet d\mathbf{r}$ [when negative, the magnitude is the amount of work needed to make it move]. This is independent of the actual speed at which the move is made.

The **total energy** thus extracted (or, with a minus sign, the work needed) when a particle moves over a segment $C$ is, *symbolically*,

$$\int_C \mathbf{g}(\mathbf{r}) \bullet d\mathbf{r}$$

or

$$\int_C g_1 dx + \ g_2 dy + \ g_3 dz$$

(alternate notation) and can be computed by parametrizing the curve (any way we like – the result is independent of the parametrization, i.e. the actual motion of the particle) and finding

$$\int_a^b \mathbf{g}[\mathbf{r}(t)] \bullet \dot{\mathbf{r}}(t)\, dt$$

EXAMPLE: Evaluate $\int_C (5z, xy, x^2 z) \bullet d\mathbf{r}$ where $C \equiv (t, t, t^2)$, $t \in (0, 1)$.

Solution: $\int_0^1 (5t^2, t^2, t^4) \bullet (1, 1, 2t)\, dt \ =$ $\int_0^1 (6t^2 + 2t^5)\, dt = \ \frac{7}{3} = \ 2.3333.$

Note that, in general, the integral is *path* *dependent*, i.e. connecting the same two points by a different curve results in two different answers.

EXAMPLE: Compute the same $\int\limits_C (5z, xy, x^2 z) \bullet d\mathbf{r}$, where now $C \equiv (t, t, t)$, $t \in (0, 1)$.

Solution: $\int\limits_0^1 (5t, t^2, t^3) \bullet (1, 1, 1)\, dt = \int\limits_0^1 (5t + t^2 + t^3)\, dt = \frac{37}{12} = 3.0833$.

Is there a *special type* of vector fields to make all such vector integrals **Path Independent?**

The answer is yes, this happens for any $\mathbf{g}$ which can be written as

$$\boldsymbol{\nabla} f(x, y, z)$$

[a gradient of a scalar field $f$, which is called the corresponding potential; $\mathbf{g}$ is then called a conservative vector field].

Proof: $\int\limits_C (\boldsymbol{\nabla} f) \bullet d\mathbf{r} = \int\limits_a^b (\boldsymbol{\nabla} f[\mathbf{r}(t)]) \bullet$

$\dot{\mathbf{r}}(t)\,dt = [\leftarrow \text{chain rule}] \int\limits_a^b \frac{df[\mathbf{r}(t)]}{dt} dt = f[\mathbf{r}(b)] - f[\mathbf{r}(a)]$.

But how can we establish whether a given $\mathbf{g}$ is conservative? Easily, the sufficient and necessary **condition** is

$$\mathsf{Curl}(\mathbf{g}) \equiv \mathbf{0}$$

Proof: $\mathbf{g} = \nabla f$ clearly implies that $\mathsf{Curl}(\mathbf{g}) \equiv \mathbf{0}$.

Now the reverse: Given such a $\mathbf{g}$, we construct (as discussed in the subsequent example) $f = \int g_1 dx + \int g_2 dy - \int \left( \int \frac{\partial g_1}{\partial y} dx \right) dy + \int g_3 dz - \int \left( \int \frac{\partial g_1}{\partial z} dx \right) dz - \int \left( \int \frac{\partial g_2}{\partial z} dy \right) dz + \int \left[ \int \left( \int \frac{\partial^2 g_1}{\partial y \partial z} dx \right) dy \right] dz$.

This implies: $\frac{\partial f}{\partial x} = g_1 + \int \frac{\partial g_1}{\partial y} dy - \int \frac{\partial g_1}{\partial y} dy + \int \frac{\partial g_1}{\partial z} dz - \int \frac{\partial g_1}{\partial z} dz - \int \left( \int \frac{\partial^2 g_1}{\partial y \partial z} dy \right) dz + \int \left[ \int \frac{\partial^2 g_1}{\partial y \partial z} dy \right] dz \equiv g_1$.

Similarly, we can show $\frac{\partial f}{\partial y} = g_2$ and $\frac{\partial f}{\partial z} = g_3$.

Note that when $\mathbf{g}$ is conservative, all we need to specify is the starting and final point of the arc (how you connect them is irrelevant, as long as you avoid an occasional singularity). We can then use the following **notation**:

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{g}(\mathbf{r}) \bullet d\mathbf{r}$$

which gives you a strong hint that $\mathbf{g}$ is conservative (the notation would not make sense otherwise).

EXAMPLE: Evaluate $\int_{(0,0,0)}^{(1,\frac{\pi}{4},2)} 2xyz^2 dx + [x^2 z^2 + z\cos(yz)]dy + [2x^2 yz + y\cos(yz)]dz$.

Solution: This is what we used to call 'exact differential form', extended to three independent variables. We solve it by integrating $g_1$ with respect to $x$ [calling the result $f_1$], adding $g_2 - \frac{\partial f_1}{\partial y}$ integrated with respect to $y$ [call the overall answer $f_2$], then adding the $z$ integral of $g_3 - \frac{\partial f_2}{\partial z}$, to get the final $f$. In our case, this yields

$x^2y\,z^2$ for $f_1$, $x^2yz^2 + \sin(yz)$ for $f_2 \equiv f$, as nothing is added in the last step. Thus $f(x, y, z) = x^2yz^2 + \sin(yz)$ [check].
Answer: $f(1, \frac{\pi}{4}, 2) - f(0, 0, 0) = 1 + \pi = 4.1416$.

Optional: We mention in passing that, similarly, $\mathsf{Div}(\mathbf{g}) = 0 \Leftrightarrow$ there is a vector field $\mathbf{h}$ say such that $\mathbf{g} \equiv \mathsf{Curl}(\mathbf{h})$ [$\mathbf{g}$ is then called purely rotational]. Any vector field $\mathbf{g}$ can be written as $\mathsf{Grad}(f) + \mathsf{Curl}(\mathbf{h})$, i.e. decomposed into its conservative and purely rotational part.

## DOUBLE INTEGRALS

can be evaluated by two consecutive (univariate) integrations, the first with respect to $x$, over its *conditional* range given $y$, the second with respect to $y$, over its *marginal* range (or the other way round, the two answers must agree).

EXAMPLES:

- To integrate over the $\begin{cases} x > 0 \\ y > 0 \\ x + y < 1 \end{cases}$ triangle, we first do $\int\limits_0^{1-y} ...\, dx$ followed by $\int\limits_0^1 ...\, dy$ (or $\int\limits_0^{1-x} ...\, dy$ followed by $\int\limits_0^1 ...\, dx$).
- To integrate over $0 < y < \frac{1}{x}$, where $1 < x < 3$, we can do either $\int\limits_1^3 \int\limits_0^{\frac{1}{x}} ...\, dy\, dx$ or $\int\limits_{\frac{1}{3}}^1 \int\limits_1^{\frac{1}{y}} ...\, dx\, dy + \int\limits_0^{\frac{1}{3}} \int\limits_1^3 ...\, dx\, dy$ [only a graph of the region can reveal why it is so].
- $\iint\limits_{x^2+y^2<1} y^2\, dx\, dy = \int\limits_{-1}^1 \left( \int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2\, dx \right) dy =$ $\int\limits_{-1}^1 2y^2\sqrt{1-y^2} dy =$ $\left[\frac{1}{4}\arcsin y + \frac{1}{4} y\sqrt{1-y^2} - 12y(1-y^2)^{\frac{3}{2}}\right]_{y=-1}^1 =$

$\frac{\pi}{4}$.

The last of these double integrals can be simplified by introducing

**Polar Coordinates**

(effectively a change of variables) by:

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}$$

One has to remember that $dx\,dy$ of the double integration must be replaced by $dr\,d\varphi$, further *multiplied* by the **Jacobian** of the transformation, namely the absolute value of

$$\left| \begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial y}{dr} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{matrix} \right|$$

In our case (of polar coordinates) this equals to $\left| \begin{matrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{matrix} \right| = r$.

EXAMPLE: $\iint\limits_{x^2+y^2<1} y^2\,dx\,dy = \int\limits_0^{2\pi}\int\limits_0^1 r^2\sin^2\varphi \cdot$

$r\,dr\,d\varphi = \int\limits_0^1 r^3 dr \times \int\limits_0^{2\pi} \sin^2\varphi\,d\varphi =$

$\frac{1}{4} \times \left[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4}\right]_0^{2\pi} = \frac{\pi}{4}$

Similarly to polar coordinates, one can introduce any other set of **new variables** to simplify the integration (the actual form of the transformation would be normally suggested to you).

EXAMPLE: $\iint\limits_{\mathcal{R}} y^2\, dx\, dy$ where $\mathcal{R}$ is a square with corners at $(0,1)$, $(1,0)$, $(0,-1)$ and $(-1,0)$.

Introducing $u, v$ by $x = u + v$ and $y = u - v$, we will cover the same square with $\frac{-1}{2} < u < \frac{1}{2}$ and $\frac{-1}{2} < v < \frac{1}{2}$. Furthermore, the Jacobian of this transformation equals to 2.

Solution: $\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} (u - v)^2\, du\, dv = 2 \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{u^3}{3} - 2\frac{u^2}{2}v + uv\right]_{u=-\frac{1}{2}}^{\frac{1}{2}} dv = 2 \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{12} + v^2)\, dv = 2(\frac{1}{12} + \frac{1}{12}) = \frac{1}{3}$.

An important **special case** is integrating a *constant*, say $c$, which can often be done geometrically. i.e.

$$\iint\limits_{\mathcal{R}} c\,dx\,dy = c \cdot Area(\mathcal{R})$$

**Applications**

of two-dimensional integrals to geometry and physics:

An **area** of a 2-D region $\mathcal{R}$ is computed by

$$\iint\limits_{\mathcal{R}} dx\,dy$$

**Center of mass** of a 2-D object (lamina) is computed by

$$\frac{\iint x\rho(x,y)\,dx\,dy}{\iint \rho(x,y)\,dx\,dy}$$

[$x$ component] and

$$\frac{\iint y\rho(x,y)\,dx\,dy}{\iint \rho(x,y)\,dx\,dy}$$

[$y$ component], where $\rho(x,y)$ is the corresponding *mass density*. When the object is of uniform density ($\rho \equiv const.$), the formulas simplify to

$$\frac{\iint x\,dx\,dy}{\iint dx\,dy}$$

and

$$\frac{\iint y\,dx\,dy}{\iint dx\,dy}$$

**Moment of inertia** with respect to some axis (this is needed when computing angular acceleration as torque/moment-of-inertia):

$$\iint d(x,y)^2 \cdot \rho(x,y)\,dx\,dy$$

where $d(x,y)$ is the (perpendicular) distance of $(x,y)$ from the axis [when the axis is $x$, $d \equiv y$ and vice versa; when the axis is $z$, $d = \sqrt{x^2+y^2}$].

**3-D volume**

$$\iint h(x,y)\,dx\,dy$$

where $h(x,y)$ is the object's 'thickness' (height) at $(x,y)$.

EXAMPLES:

1. Find the center of mass of a half disk of

radius $R$ and uniform mass density.

Solution: We position the object in the upper half plane with its center at the origin, and use polar coordinates to evaluate: $\frac{\int_0^\pi \int_0^R r \sin\varphi \cdot r \, dr \, d\varphi}{\int_0^\pi \int_0^R r \, dr \, d\varphi} = \frac{\frac{R^3}{3} \cdot (\cos 0 - \cos\pi)}{\frac{R^2}{2} \cdot \pi} = \frac{4R}{3\pi} = 0.42441R$ [its $y$ component]. From symmetry, its $x$ component must be equal to zero.

2. Find the volume of a cone with circular base of radius $R$ and height $H$.

   We do this in polar coordinates where the formula for $h(r, \varphi)$ simplifies to $H \cdot \frac{R-r}{R}$.

   Answer: $\frac{H}{R} \int_0^{2\pi} \int_0^R (R - r) \cdot r \, dr \, d\varphi = \frac{2\pi H}{R} \cdot [R\frac{r^2}{2} - \frac{r^3}{3}]_{r=0}^R = \frac{\pi R^2 H}{3}$ (check).

3. Find the volume of a sphere of radius $R$.

   Solution: Introducing polar coordinates in $x, y$, the $z$-thickness is $h(x, y) = 2\sqrt{R^2 - r^2}$ [Pythagoras]. Integrating this over the sphere's $x, y$ projection (a circle of radius $R$) yields $2\int\limits_0^{2\pi}\int\limits_0^{R}\sqrt{R^2-r^2}\cdot$ $r\,dr\,d\varphi = 4\pi\left[-\frac{1}{3}(R^2-r^2)^{\frac{3}{2}}\right]_{r=0}^{R} =$ $\frac{4}{3}\pi R^3$ (check).

4. Find the volume of the (solid) cylinder $x^2+z^2<1$ cut along $y=0$ and $z=y$ [i.e. $0<y<z$].

   Solution: Its $x,z$ projection is a half-circle $x^2+z^2<1$ with $z>0$, its thickness along $y$ is $h(x,z)=z$. Replacing $x$ and $z$ by polar coordinates, we can readily integrate $\int\limits_0^{\pi}\int\limits_0^{R} r\sin\varphi\cdot r\,dr\,d\varphi =$ $\frac{1}{3}\cdot[-\cos\varphi]_{\varphi=0}^{\pi} = \frac{2}{3}$. There are two alternate ways of computing the volume, integrating the $z$-thickness over the $(x,y)$ projection, or the $x$-thickness over $dy\,dz$ [try both of them].

5. Find the volume of the 3-D region defined by $x^2+y^2<1$ and $y^2+z^2<1$ [the common part of two cylinders crossing

each other at the right angle].

Solution: The $(x, y)$ projection of the region is describe by $x^2 + y^2 < 1$ (now a *circle*, not a cylinder), the corresponding $z$-thickness is $h(x, y) = 2\sqrt{1 - y^2}$.

Answer: $2 \int\limits_0^{2\pi} \int\limits_0^1 \sqrt{1 - r^2 \sin^2 \varphi} \cdot$

$$r\, dr\, d\varphi = \int\limits_0^{2\pi} -\frac{2}{3\sin^2\varphi} \left[ (1 - r^2 \sin^2 \varphi)^{\frac{3}{2}} \right]_{r=0}^{1} d\varphi =$$

$$\int\limits_0^{2\pi} \frac{2(1-|\cos\varphi|^3)}{3\sin^2\varphi}\, d\varphi = \int\limits_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{2(1-|\cos\varphi|^3)}{3\sin^2\varphi}\, d\varphi =$$

$$\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2(1-\cos^3\varphi)}{3\sin^2\varphi}\, d\varphi + \int\limits_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{2(1+\cos^3\varphi)}{3\sin^2\varphi}\, d\varphi = \frac{16}{3}.$$

[The integration is quite tricky, later on we learn how to deal with it more efficiently].

An alternate way is to use the $(x, z)$ projection (a unit square, divided by its two diagonals into four sections of identical volume), and then integrate over one of these sections (say the right-most) the corresponding $y$-

thickness $h(x,z) = 2\sqrt{1-x^2}$, thus: $2\int\limits_0^1 \sqrt{1-x^2} \int\limits_{-x}^{x} dz\, dx = 2\int\limits_0^1 \sqrt{1-x^2} \cdot 2x\, dx = -\frac{4}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_{x=0}^{1} = \frac{4}{3}$. The total volume is four times bigger (check). The integration was now a lot easier.

In these type of questions, it is important to first identify each *side* of the 3-D object (and the corresponding equation), and each of its *edges* (described by two equations). To *project* a specific edge into, say, the $(x,y)$ plane, one must eliminate $z$ from one of the two equations and substitute into the other (getting a single $x$-$y$ equation).

## Surfaces in 3-D

There are two ways of **defining** a 2-D surface:

1. By an **equation**: $f(x,y,z) = c$ [$c$ being a constant].
2. **Parametrically**: $\mathbf{r}(u,v) \equiv [x(u,v), y(u,v), z(u,v)]$

(three arbitrary functions of two parameters $u$ and $v$; restricting these to a 2-D region selects a *section* of the surface). ■

EXAMPLES:

- Parametrize a *sphere* of radius $a$.

  Answer: $\mathbf{r}(u,v) = [a \sin v \cos u, a \sin v \sin u, a \cos v]$ where $0 \leq u < 2\pi$ and $0 \leq v \leq \pi$ [later on we introduce the so called *spherical coordinates* in almost the same manner – they are usually called $r$, $\theta$ and $\varphi$ rather than $a$, $v$ and $u$]. The curves we get by fixing $v$ and varying $u$ (or vice versa) are called *'coordinate' curves* [latitude circles and longitude half-circles in this case].

- Identify $\mathbf{r}(u,v) = [u \cos v, u \sin v, u]$.

  Answer: a $45^o$ *cone* centered on $z$.

- Parametrize the *cylinder* $x^2 + y^2 = a^2$.

  Solution: $\mathbf{r}(u,v) = [a \cos u, a \sin u, v]$.

- Identify $[u \cos v, u \sin v, u^2]$.

  Answer: A *paraboloid* centered on $+z$.

## Surface integrals

Let us consider a **specific parametrization** of a surface. It is obvious that $\frac{\partial \mathbf{r}}{\partial u}$ [componentwise operation, keeping $v$ fixed] is a *tangent direction* to the corresponding coordinate curve and consequently tangent to the surface itself. Similarly, so is $\frac{\partial \mathbf{r}}{\partial v}$ (note that these two don't have to be orthogonal). Constructing the corresponding *tangent plane* is then quite trivial.

Consequently,

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

yields a direction *normal* (perpendicular) to the surface, and its magnitude $\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|$, multiplied by $du\,dv$, provides the *area* of the corresponding (infinitesimal) *parallelogram*, obtained by increasing $u$ by $du$ and $v$ by $dv$ [$\frac{\partial \mathbf{r}}{\partial u}du$ and $\frac{\partial \mathbf{r}}{\partial v}dv$ being its two sides]. This can be seen from:

$$\left|\frac{\partial \mathbf{r}}{\partial u}du \times \frac{\partial \mathbf{r}}{\partial v}dv\right| = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u}\right| du\,dv \equiv dA$$

Since $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2 \gamma = |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2 \gamma) = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \bullet \mathbf{b})^2$, we can simplify it to

$$dA = \sqrt{\left|\frac{\partial \mathbf{r}}{\partial u}\right|^2 \left|\frac{\partial \mathbf{r}}{\partial v}\right|^2 - \left(\frac{\partial \mathbf{r}}{\partial u} \bullet \frac{\partial \mathbf{r}}{\partial v}\right)^2}\, du\, dv$$

which is more convenient computationally (bypassing the cross product).

To find an **area** of a whole **surface** (or its section), we need to 'add' the contributions from all these parallelograms, thus:

$$\text{Area} = \iint_{\mathcal{S}} dA = \iint_{\mathcal{R}} \sqrt{\left|\frac{\partial \mathbf{r}}{\partial u}\right|^2 \left|\frac{\partial \mathbf{r}}{\partial v}\right|^2 - \left(\frac{\partial \mathbf{r}}{\partial u} \bullet \frac{\partial \mathbf{r}}{\partial v}\right)^2}\, du\, dv$$

where $\mathcal{R}$ is the $(u, v)$ region needed to cover the (section of the) surface $\mathcal{S}$. Needless to say, the answer must be the same, *regardless* of parametrization.

EXAMPLES:

1. Find the tangent plane to the *ellipsoid* $3x^2 + 2y^2 + z^2 = 20$ at $(1, 2, 3)$.

Solution: First one can easily check that the point is on the ellipsoid (just in case). We can parametrize the *upper half* of the ellipsoid (which is sufficient in this case) by $\mathbf{r}(u, v) = (u, v, \sqrt{20 - 3u^2 - 2v^2})$. Then $\frac{\partial \mathbf{r}}{\partial u} = (1, 0, -\frac{3u}{\sqrt{20-3u^2-2v^2}}) = (1, 0, -1)$ and $\frac{\partial \mathbf{r}}{\partial v} = (0, 1, -\frac{2v}{\sqrt{20-3u^2-2v^2}}) = (0, 1, -\frac{4}{3})$. The corresponding cross product $(1, 0, -1) \times (0, 1, -\frac{4}{3}) = (1, \frac{4}{3}, 1)$ yields the tangent plane's normal; we also know that the plane has to pass through $(1, 2, 3)$.

Answer: $3x + 4y + 3z = 20$.

2. Find the area of a surface of a *sphere* of radius $a$.

Solution: Using the

$$\mathbf{r}(u, v) = (a \sin v \cos u,\ a \sin v \sin u,\ a \cos v)$$

parametrization, we get:

$$\frac{\partial \mathbf{r}}{\partial u} = (-a \sin v \sin u,\ a \sin v \cos u,\ 0)$$

and

$$\frac{\partial \mathbf{r}}{\partial v} = (a\cos v\cos u,\ a\cos v\sin u,\ -a\sin v)$$
$\Rightarrow\ dA \equiv\ a^2\,|\sin v|\ du\,dv.$

Answer: $a^2 \int\limits_0^{2\pi} \int\limits_0^{\pi} \sin v\,dv\,du =\ 4\pi a^2.$

3. Find the surface area of a *torus* (donut) of dough-radius equal to $b$ and hole-radius equal to $a-b$.

Solution: We make $z$ its axis, and $[0, a+b\cos v, b\sin v]$ its cross section with the $(y,z)$ plane. The full parametrization is then: $\mathbf{r}(u,v) =\ [(a+b\cos v)\cos u,\ (a+b\cos v)\sin u,\ b\sin v]$, where both $u$ and $v$ vary from $0$ to $2\pi$. This yields $\frac{\partial \mathbf{r}}{\partial u} = [-(a+b\cos v)\sin u,\ (a+b\cos v)\cos u, 0]$, $\frac{\partial \mathbf{r}}{\partial v} = [-b\sin v\cos u,\ b\sin v\sin u,\ -b\cos v] \Rightarrow dA \equiv b(a+b\cos v).$

Answer: $b \int\limits_0^{2\pi} \int\limits_0^{2\pi} (a+\cos v)\,dv\,du\ = 4\pi^2 ab.$

Computing areas is just a special case of a

**Surface Integral of Type I**

('*scalar*' type). In general, we can integrate any scalar function $f(x,y,z)$ over a surface $\mathcal{S}$ [symbolic **notation** $\iint\limits_{\mathcal{S}} f(x,y,z)\,dA$] by parametrizing the surface and computing

$$\iint\limits_{\mathcal{R}} f[\mathbf{r}(u,v)]\cdot\sqrt{\left|\frac{\partial \mathbf{r}}{\partial u}\right|^2\left|\frac{\partial \mathbf{r}}{\partial v}\right|^2-\left(\frac{\partial \mathbf{r}}{\partial u}\bullet\frac{\partial \mathbf{r}}{\partial v}\right)^2}\,du\,dv$$

[the answer is independent of parametrization].

When divided by the corresponding surface *area*, this represents the **average** of $f(x,y,z)$ over $\mathcal{S}$.

**Other applications** to Physics are:

1. **Moment of inertia** of a shell-like structure (*lamina*) of surface *density* $\rho(x,y,z)$:
$$\iint\limits_{\mathcal{S}} d^2\cdot\rho\cdot dA$$

where $d(x, y, z)$ is the distance from the rotation axis. For a lamina of *uniform* density, $\rho = \frac{M}{A}$ (total mass over total area).

2. **Center of mass**

$$\left[\frac{\iint\limits_{\mathcal{S}} x \cdot \rho \cdot dA}{\iint\limits_{\mathcal{S}} \rho \cdot dA}, \frac{\iint\limits_{\mathcal{S}} y \cdot \rho \cdot dA}{\iint\limits_{\mathcal{S}} \rho \cdot dA}, \frac{\iint\limits_{\mathcal{S}} z \cdot \rho \cdot dA}{\iint\limits_{\mathcal{S}} \rho \cdot dA}\right]$$

($\rho$ cancels out when constant, i.e. uniform mass density). Note that $\iint\limits_{\mathcal{S}} \rho \cdot dA$ is the total mass.

EXAMPLE: Find the moment of inertia of a spherical shell of radius $a$ and total mass $M$ (uniformly distributed) with respect to an axis going through its center.

Solution: 'Borrowing' the parametrization (and $dA$) from the previous Example $2$, and using $z$ as the axis, we get $\rho \iint\limits_{\mathcal{S}} (x^2 + y^2)\, dA = \rho \int\limits_0^{\pi} \int\limits_0^{2\pi} a^2 \sin^2 v \cdot a^2 \sin v \, du \, dv = 2\pi\rho a^4 [\frac{\cos^3 v}{3} - \cos v]_{v=0}^{\pi} = 2\pi \frac{M}{4\pi a^2} a^4 \cdot \frac{4}{3} =$

$\frac{2}{3}Ma^2$.

## Surface integrals of Type II

('*vector*' type): When integrating a vector field $\mathbf{g}(x, y, z)$ [representing some stationary flow] over an *orientable* (having two sides) surface $\mathcal{S}$, we are usually interested in computing the *total flow* (**flux**) through this surface, in a chosen direction.

The flow through an '*infinitesimal*' area [our parallelogram] of the surface is given by the dot product

$$\mathbf{g} \bullet \mathbf{n}\, dA$$

where $\mathbf{n}$ is a unit direction *normal* [perpendicular] to the area, since the flow is obviously proportional to the area's size $dA$, to the magnitude of $\mathbf{g}$ (the flow's speed), *and* to the cosine of the $\mathbf{n}$-$\mathbf{g}$ angle.

’Adding’ these, one gets

$$\iint\limits_{\mathcal{S}} \mathbf{g} \bullet \mathbf{n} \, dA \equiv \iint\limits_{\mathcal{S}} \mathbf{g} \bullet d\mathbf{A}$$

$$\equiv \iint\limits_{\mathcal{S}} (g_1 \, dy \, dz + \, g_2 \, dz \, dx + \, g_3 \, dx \, dy)$$

introducing two more alternate, *symbolic notations* (I usually use the middle one).

We can convert this to a regular **double-integral** (in $u$ and $v$), by *parametrizing* the surface [different parametrizations must give the same correct answer] and replacing $\mathbf{n} \, dA$ by $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ [having both the correct *area* and *direction*], getting:

$$\iint\limits_{\mathcal{R}} \mathbf{g}[\mathbf{r}(u,v)] \bullet \left[ \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right] \, du \, dv$$

where $\mathcal{R}$ is the $(u,v)$ region corresponding to $\mathcal{S}$. Note that $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ does not necessarily have the correct (originally prescribed) *orientation*; when that happens, we fix it by reversing the sign of the result.

EXAMPLES (to simplify our notation, we use $\frac{\partial \mathbf{r}}{\partial u} \equiv \mathbf{r}_u$ and $\frac{\partial \mathbf{r}}{\partial v} \equiv \mathbf{r}_v$):

1. Evaluate $\iint\limits_{\mathcal{S}} (x, y, z-3) \bullet d\mathbf{A}$ where $\mathcal{S}$ is the upper (i.e. $z > 0$) half of the $x^2 + y^2 + z^2 = 9$ sphere, oriented upwards.

   Solution: Here we can bypass spherical coordinates (why?) and use instead $\mathbf{r}(u,v) = \left[u, v, \sqrt{9-u^2-v^2}\right]$ with $u^2 + v^2 < 9$ [defining the two-dimensional region $\mathcal{R}$ over which we integrate]. Furthermore, $\mathbf{r}_u = [1, 0, -\frac{u}{\sqrt{9-u^2-v^2}}]$ and $\mathbf{r}_v = [0, 1, -\frac{v}{\sqrt{9-u^2-v^2}}] \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = [\frac{u}{\sqrt{9-u^2-v^2}}, \frac{v}{\sqrt{9-u^2-v^2}}, 1]$ [correct orientation!] $\Rightarrow \mathbf{g} \bullet (\mathbf{r}_u \times \mathbf{r}_v) = \frac{u^2+v^2}{\sqrt{9-u^2-v^2}} + \sqrt{9-u^2-v^2} - 3 = \frac{9}{\sqrt{9-u^2-v^2}} - 3$. The actual integration will be done in polar coordinates: $\int\limits_0^{2\pi} \int\limits_0^3 \left(\frac{9}{\sqrt{9-r^2}} - 3\right) r\, dr\, d\varphi = 2\pi \left[-9\sqrt{9-r^2} - 3\frac{r^2}{2}\right]_{r=0}^3 = 27\pi.$

2. Evaluate $\iint\limits_{\mathcal{S}} (yz, xz, xy) \bullet d\mathbf{A}$ where $\mathcal{S}$ is the full $x^2 + y^2 + z^2 = 1$ sphere oriented outwards. Using the usual parametrization: $\mathbf{r}(u, v) = (\cos u \sin v,\ \sin u \sin v,\ \cos v)$ $\Rightarrow \mathbf{r}_u = (-\sin u \sin v, \cos u \sin v, 0)$, $\mathbf{r}_v = (\cos u \cos v, \sin u \cos v, -\sin v)$ and $\mathbf{r}_u \times \mathbf{r}_v = (-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v)$ [wrong orientation, reverse its sign!], we get $\mathbf{g} \bullet (-\mathbf{r}_u \times \mathbf{r}_v) = 3 \cos u\ \sin u\ \sin^3 v\ \cos v$.

Answer: $3 \int\limits_0^{2\pi} \sin u \cos u \, du \times \int\limits_0^{\pi} \sin^3 v\ \cos v \, dv = 0$

Shortly we learn a shortcut for evaluating Type II integrals over a *closed* surface which will make the last example trivial. But first we need to discuss

**'Volume' (triple) integral**

is a 3-D generalization of a 2-D integrals

$$\iiint_{\mathcal{V}} f(x, y, z)\, dV$$

and is evaluated by converting it to three consecutive univariate integrations:

$$\int_{L_z}^{U_z} \int_{L_y(z)}^{U_y(z)} \int_{L_x(y,z)}^{U_x(y,z)} f(x, y, z)\, dx\, dy\, dz$$

When the region of integration is a sphere, we can simplify the integration by introducing **spherical coordinates** $r$, $\theta$ and $\varphi$, by

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta \end{aligned}$$

This means that $dx\, dy\, dz$ ($\equiv dV$) needs to be replaced by $dr\, d\theta\, d\varphi$ *multiplied* by the corresponding Jacobian, namely:

$$\begin{vmatrix} \sin\theta \cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} =$$

$$r^2 \cos^2\theta \sin\theta + r^2 \sin^3\theta =$$

$$r^2 \sin\theta$$

Similarly to double integration of a

**constant**,
$$\iiint\limits_{\mathcal{V}} c\,dV = c \cdot \text{Volume}(\mathcal{V})$$
whenever we remember a formula for the corresponding volume.

**Possible Applications**

of triple integrals include computing **volume** of a 3-D body
$$V = \iiint\limits_{\mathcal{V}} dV$$
**averaging** a scalar function $f(x, y, z)$ over a 3-D region
$$\frac{\iiint\limits_{\mathcal{V}} f(x, y, z)\,dV}{V}$$
computing the **center of mass** of a 3-D object of mass density $\rho(x, y, z)$ [it cancels out when constant]: $\left[\frac{\iiint\limits_{\mathcal{V}} x\,\rho(x,y,z)\,dV}{\iiint\limits_{\mathcal{V}} \rho(x,y,z)\,dV}, \frac{\iiint\limits_{\mathcal{V}} y\,\rho(x,y,z)\,dV}{\iiint\limits_{\mathcal{V}} \rho(x,y,z)\,dV}, \frac{\iiint\limits_{\mathcal{V}} z\,\rho(x,y,z)\,dV}{\iiint\limits_{\mathcal{V}} \rho(x,y,z)\,dV}\right]$

and computing the corresponding **moment of**

**inertia**

$$\iiint\limits_{\mathcal{V}} d^2\,\rho\,dV$$

where $d(x, y, z)$ is distance from the rotational axis, and $\rho \equiv \frac{M}{V}$ when the mass density is uniform.

EXAMPLE: Find the moment of inertia of a uniform *sphere* of radius $a$ with an axis going through its center.

Solution: $\frac{M}{\frac{4}{3}\pi a^3} \iiint\limits_{\mathcal{V}} (x^2 + y^2) dV =$

$\frac{M}{\frac{4}{3}\pi a^3} \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^{a} r^2 \sin^2\theta \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi =$

$\frac{M}{\frac{4}{3}\pi a^3} \cdot \frac{a^5}{5} \cdot \left[\frac{\cos^3\theta}{3} - \cos\theta\right]_{\theta=0}^{\pi} \cdot 2\pi = \frac{2}{5} M a^2.$

There is an interesting and **useful relationship** between a Type II integral over a closed (outward oriented) surface $\mathcal{S}_c$, and a volume integral over the 3-D region $\mathcal{V}$ enclosed by this $\mathcal{S}_c$, called

**Gauss Theorem**

$$\iint\limits_{\mathcal{S}_c} \mathbf{g} \bullet \, d\mathbf{A} \equiv \iiint\limits_{\mathcal{V}} \mathrm{Div}(\mathbf{g})\, dV$$

provided that $Div(\mathbf{g})$ has no singularities throughout $\mathcal{V}$.

Proof: If it's true for an 'infinitesimal' cube, it is true for 3-D region of any size.

For an infinitesimal cube (of size $h^3$) we get, for the $x - \frac{h}{2}$ and $x + \frac{h}{2}$ sides:

$$\mathbf{g}(x - \tfrac{h}{2}, y, z) \simeq \mathbf{g}(x, y, z) - \tfrac{h}{2}\frac{\partial \mathbf{g}(x, y, z)}{\partial x}$$

$$\mathbf{g}(x + \tfrac{h}{2}, y, z) \simeq \mathbf{g}(x, y, z) + \tfrac{h}{2}\frac{\partial \mathbf{g}(x, y, z)}{\partial x}$$

They are to be 'dot' multiplied by $(-h^2, 0, 0)$ and $(h^2, 0, 0)$ respectively, and added, getting $h^3\frac{\partial g_1(x,y,z)}{\partial x}$. The other 4 sides wouldsimilarly contribute $h^3\frac{\partial g_2(x,y,z)}{\partial y}$ and $h^3\frac{\partial g_3(x,y,z)}{\partial z}$. All together, this results in $h^3\mathrm{Div}(\mathbf{g})$, which is what we have on the right hand side.

EXAMPLES:

1. The integral of Example 2 from the

previous section thus becomes quite trivial, as $\mathrm{Div}([yz, xz, xy]) \equiv 0$.

2. Evaluate $\iint\limits_{\mathcal{S}} (x^3, x^2y, x^2z) \bullet \mathbf{n}\, dA$, where $\mathcal{S}$ is the surface of $\begin{cases} x^2+y^2<a^2 \\ 0<z<b \end{cases}$ (a cylinder of radius $a$ and height $b$), oriented outwards.

Solution: Using the Gauss theorem, we get $\iiint\limits_{\substack{x^2+y^2<a^2 \\ 0<z<b}} 5x^2\, dV = 5b \iint\limits_{x^2+y^2<a^2} x^2\, dx\, dy =$ $5b \int\limits_0^{2\pi} \int\limits_0^a r^2 \cos^2\varphi \cdot r\, dr\, d\varphi$ [going polar] $= 5b^2 \cdot \frac{a^4}{4} \cdot \pi = \frac{5}{4} a^4 b\pi.$

Let us *verify* this by recomputing the original surface integral directly (note that now we have to deal with three distinct surfaces: the top disk, the bottom disk, and the actual cylindrical walls): The top can be parametrized by $\mathbf{r}(u,v) = [u, v, b]$, contributing $\iint\limits_{u^2+v^2<a^2} [u^3, u^2v, u^2b] \bullet (0,0,1)\, du\, dv =$

$b\int\limits_0^{2\pi}\int\limits_0^{a} r^2\cos^2\varphi\cdot r\,dr\,d\varphi$ [polar] $= b\frac{a^4}{4}\pi$. The bottom is parametrized by $\mathbf{r}(u,v) = [u,v,0]$, contributing minus (because of the wrong orientation) $\iint\limits_{u^2+v^2<a^2} [u^3, u^2v, 0] \bullet (0,0,1)\,du\,dv \equiv 0$. Finally, the sides are parametrized by $\mathbf{r}(u,v) = [a\cos u, a\sin u, v]$, contributing $\iint\limits_{\substack{0<u<2\pi \\ 0<v<b}} [a^3\cos^3 u, a^3\cos^2 u\sin u, a^2v\cos^2 u] \bullet [a\cos u, a\sin u, 0]\,du\,dv = a^4\int\limits_0^{b}\int\limits_0^{2\pi}\cos^2 u\,du\,dv = a^4b\pi$. Adding the three contributions gives $\frac{5}{4}a^4b\pi$ [check].

Similarly, there is an **interesting relationship** between the Type II *line* integral over a *closed* curve $\mathcal{C}_d$ and a Type II *surface* integral over *any* surface $\mathcal{S}$ having $\mathcal{C}_d$ as its boundary, called

**Stokes' Theorem**

$$\iint_{\mathcal{S}} \mathrm{Curl}(\mathbf{g}) \bullet \mathbf{n}\, dA \equiv \oint_{\mathcal{C}_d} \mathbf{g} \bullet\ d\mathbf{r}$$

where the orientation of $\mathcal{C}_d$ and that of $\mathbf{n}\, dA$ follow the right-handed pattern. [When $\mathcal{C}_d$ and $\mathcal{S}$ lie in the $(x, y)$ plane, this is known as the Green's Theorem].

Proof: Similarly, we can divide the surface into many infinitesimal 'near' squares of size $h^2$ We need to prove that the above relationship holds for each of them. Furthermore, we can always use coordinate system which makes the square lie (sides alligned with axes) to $x$-$y$ plane.

On the left hand side, we would thus get

$$h^2 \left( \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \right)$$

On the RHS, considering the $x - \frac{h}{2}$ and $x + \frac{h}{2}$

sides,we get the same

$$\mathbf{g}(x - \tfrac{h}{2}, y, z) \simeq \mathbf{g}(x, y, z) - \tfrac{h}{2}\frac{\partial \mathbf{g}(x, y, z)}{\partial x}$$

$$\mathbf{g}(x + \tfrac{h}{2}, y, z) \simeq \mathbf{g}(x, y, z) + \tfrac{h}{2}\frac{\partial \mathbf{g}(x, y, z)}{\partial x}$$

as before. They are to be dot multiplied by $(0, -h, 0)$ and $(0, h, 0)$ respectively. Together, this yields $h^2\frac{\partial g_2}{\partial x}$. Similarly, the other 2 sides contribute the remaining $-h^2\frac{\partial g_1}{\partial y}$.

EXAMPLE: Evaluate $\oint\limits_{\mathcal{C}_d} (y, xz^3, -zy^3) \bullet d\mathbf{r}$, where $\mathcal{C}_d$ is defined by $\left\{ \begin{array}{c} x^2 + y^2 = 4 \\ z = -3 \end{array} \right.$, counterclockwise when viewed from the top.

Solution: Using Stokes' Theorem we replace this integral by $\iint\limits_{\mathcal{S}} [-3zy^2 - 3xz^2,\ 0,\ z^3 - 1] \bullet \mathbf{n}\, dA$, where $\mathcal{S}$ is the corresponding (flat) disk. Parametrizing $\mathcal{S}$ by $\mathbf{r}(u, v) \equiv [u, v, -3] \Rightarrow \mathbf{n}\, dA = [0, 0, 1]\, du\, dv$, this converts to

$$\iint\limits_{u^2+v^2<4} (-28)\, du\, dv = -28 \cdot 4\pi = -112\pi$$

[note that we did not need to know the first two components of $\mathrm{Curl}(\mathbf{g})$ in this case, i.e. it pays to do the $\mathbf{n}\,dA$ *first*].

We will verify the answer by performing the original line integral, directly: $\mathbf{r}(t) = [2\cos t, 2\sin t, -3]$ is the parametrization of $\mathcal{C}_d$, which converts the integral to $\int_0^{2\pi} [2\sin t, -54\cos t, 24\sin^3 t] \bullet [-2\sin t, 2\cos t, 0]\, dt = \int_0^{2\pi} (-4\sin^2 t - 108\cos^2 t)\, dt = -112\pi$ [almost equally easily].

Unless $\mathrm{Curl}(\mathbf{g}) \equiv \mathbf{0}$, the computational simplification achieved by applying the Stokes' theorem is very limited (a far cry from the Gauss theorem). One exception is when $\mathcal{C}_d$ is a 'broken' planar curve (consisting of several segments), as we can trade *one* surface integral for *several* line integrals.

**Review exercises:**

1. Find the area of the following (truncated)

paraboloid: $\begin{cases} z = x^2 + y^2 \\ z < b \end{cases}$.

Solution: Parametrize: $\mathbf{r} = [u, v, u^2 + v^2] \Rightarrow \mathbf{r}_u = [1, 0, 2u]$ and $\mathbf{r}_v = [0, 1, 2v] \Rightarrow dA = \sqrt{(1 + 4u^2)(1 + 4v^2) - 16u^2v^2}du\,dv = \sqrt{1 + 4(u^2 + v^2)}du\,dv$. We need $\iint\limits_{u^2+v^2<b} dA =$ [going polar] $\int\limits_0^{2\pi}\int\limits_0^{\sqrt{b}} \sqrt{1 + 4r^2}\cdot r dr\,d\varphi = 2\pi\cdot \left[\frac{1}{12}(1 + 4r^2)^{\frac{3}{2}}\right]_{r=0}^{\sqrt{b}} = \frac{\pi}{6}\left[(1 + 4b)^{\frac{3}{2}} - 1\right]$.

2. Evaluate $\iint\limits_{\mathcal{S}} [y, 2, xz] \bullet \mathbf{n}\,dA$, where $\mathcal{S}$ is defined by $\begin{cases} y = x^2 \\ 0 < x < 2 \\ 0 < z < 3 \end{cases}$, and $\mathbf{n}$ is pointing in the direction of $-y$.

Solution: $\mathbf{r}(u, v) = [u, u^2, v] \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = [1, 2u, 0] \times [0, 0, 1] = [2u, -1, 0]$ (correct orientation). The integral thus converts to $\int\limits_0^3\int\limits_0^2 [u^2, 2, uv] \bullet [2u, -1, 0]\,du\,dv =$

$\int\limits_0^3 \int\limits_0^2 (2u^3 - 2)\, du\, dv = 3\left[\frac{u^4}{2} - 2u\right]_{u=0}^{2} =$
12.

3. Find $\iint\limits_{\mathcal{S}} [x^2, 0, 3y^2] \bullet \mathbf{n}\, dA$, where $\mathcal{S}$ is the $\begin{cases} x > 0 \\ y > 0 \\ z > 0 \end{cases}$ portion of the $x + y + z = 1$ plane, and $\mathbf{n}$ is pointing upwards.

   Solution: $\mathbf{r} = [u, v, 1 - u - v] \Rightarrow$ $\mathbf{r}_u \times \mathbf{r}_v = [1, 0, -1] \times [0, 1, -1] = [1, 1, 1]$ (correct orientation) $\Rightarrow \int\limits_0^1 \int\limits_0^{1-v} [u^2, 0, 3v^2] \bullet$ $[1, 1, 1]\, du\, dv = \int\limits_0^1 \int\limits_0^{1-v} (u^2 + 3v^2)\, du\, dv =$ $\int\limits_0^1 \left[\frac{u^3}{3} + 3uv^2\right]_{u=0}^{1-v} dv = \int\limits_0^1 \left[\frac{(1-v)^3}{3} + 3(1 - v)v^2\right] du =$ $\left[-\frac{(1-v)^4}{12} + 3\frac{v^3}{3} - 3\frac{v^4}{4}\right]_{v=0}^{1} = \frac{1}{3}.$

4. Parametrize a circle of radius $\rho = 5$, centered on $\mathbf{a} = [1, -2, 4]$, and normal to $\mathbf{n} = [2, 0, -3]$.

Solution: In general, a circle is parametrized by: $\mathbf{r}(t) = \mathbf{a} + \rho\mathbf{m}_1 \cos t + \rho\mathbf{m}_2 \sin t$, where $\mathbf{m}_1$ and $\mathbf{m}_2$ are unit vectors perpendicular to $\mathbf{n}$ and to each other. They can be found by taking the cross product of $\mathbf{n}$ and an arbitrary vector, then taking the cross product of the resulting vector and $\mathbf{n}$, and normalizing both, thus: $[2, 0, -3] \times [1, 0, 0] = [0, -3, 0]$ and $[0, -3, 0] \times [2, 0, -3] = [9, 0, 6] \Rightarrow \mathbf{m}_1 = [0, -3, 0] \div 3 = [0, -1, 0]$ and $\mathbf{m}_2 = [9, 0, 6] \div \sqrt{9^2 + 6^2} = \left[\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}\right]$.

Answer: $\mathbf{r}(t) = [1 + \frac{15}{\sqrt{13}} \sin t,\ -2 - 5\cos t,\ 4 + \frac{10}{\sqrt{13}} \sin t]$ where $0 \le t < 2\pi$. Subsidiary: To parametrize the corresponding disk: $\mathbf{r}(u, v) = [1 + \frac{3v}{\sqrt{13}} \sin u,\ -2 - v\cos u,\ 4 + \frac{2v}{\sqrt{13}} \sin u]$ where $0 \le u < 2\pi$ and $0 \le v < 5$.

5. Find the moment of inertia (with respect to the $z$ axis) of a shell-like torus (parametrized earlier) of uniform mass

density and total mass $M$.

Solution: Recall that $\mathbf{r}(u,v) = [(a + b\cos v)\cos u,\ (a+b\cos v)\sin u,\ b\sin v] \Rightarrow dA = b(a + b\cos v)\ du\,dv$ [done earlier] and $d^2 = (a + b\cos v)^2 \Rightarrow$
$\rho \int\limits_0^{2\pi}\int\limits_0^{2\pi}(a + b\cos v)^2 b(a + b\cos v)\,du\,dv =$
$\rho b 2\pi \int\limits_0^{2\pi}(a^3 + 3a^2 b\cos v + 3ab^2\cos^2 v + b^3\cos^3 v)\,dv = \frac{M}{4\pi^2 ab} b2\pi[2\pi a^3 + 3ab^2\pi] = M\,(a^2 + \frac{3}{2}b^2)$.

6. Repeat with a solid torus.

Solution: We replace $\mathbf{r}$ by $[(a + r\cos v)\cos u,\ (a + r\cos v)\sin u,\ r\sin v]$, where the new variables $u$, $v$ and $r$ $(0 \leq r < b)$ can be also seen as orthogonal coordinates. For any orthogonal coordinates it is easy to find the Jacobian, geometrically, by $dx\,dy\,dz \to r\,dv \cdot (a + r\cos v)\,du \cdot dr = r\,(a + r\cos v)\,du\,dv\,dr \Rightarrow \rho \int\limits_0^{b}\int\limits_0^{2\pi}\int\limits_0^{2\pi}(a +$

$r\cos v)^2 r\,(a + r\cos v)\,du\,dv\,dr =$
$\rho 2\pi^2 \int\limits_0^b r\,(2a^3 + 3ar^2)\,dr = \rho 2\pi^2 (a^3 b^2 +$
$\frac{3}{4}ab^4)$.

Similarly, the total volume is $\int\limits_0^b \int\limits_0^{2\pi} \int\limits_0^{2\pi} r\,(a+$
$r\cos v)\,du\,dv\,dr = (2\pi)^2 a \frac{b^2}{2}$.

Answer: $\frac{M}{2\pi^2 ab^2} 2\pi^2 (a^3 b^2 + \frac{3}{4}ab^4) =$
$M\,(a^2 + \frac{3}{4}b^2)$.

An alternate approach would introduce polar coordinates in the $(x, y)$-plane, use $2\sqrt{b^2 - (r-a)^2}$ for the $z$-thickness and $r^2$ for $d^2$, leading to $\rho \int\limits_0^{2\pi} \int\limits_{a-b}^{a+b} r^2 \cdot$
$2\sqrt{b^2 - (r-a)^2} \cdot r\,dr\,d\varphi = \ldots$ [verify that this leads to the same answer].

7. Consider the following solid $\begin{cases} x > 0 \\ y > 0 \\ z > 0 \\ x + y + z < 1 \end{cases}$ of uniform density. Find:

(a) Center of mass.

Solution: To find its $x$-component we need to divide $\int\limits_0^1 \int\limits_0^{1-z} \int\limits_0^{1-y-z} x\, dx\, dy\, dz = \int\limits_0^1 \int\limits_0^{1-z} \frac{(1-y-z)^2}{2} dy\, dz = \int\limits_0^1 \frac{(1-z)^3}{6} dz = \frac{1}{24}$ by the volume $\int\limits_0^1 \int\limits_0^{1-z} \int\limits_0^{1-y-z} dx\, dy\, dz = \int\limits_0^1 \int\limits_0^{1-z} (1-y-z) dy\, dz = \int\limits_0^1 \frac{(1-z)^2}{2} dz = \frac{1}{6}$.

Answer: $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$, as the $y$ and $z$-components must have the same value as the $x$-component [obvious from symmetry].

(b) Moment of inertial with respect to $[t, t, t]$ (the axis).

Solution: To find $d^2$ we project $[x, y, z]$ into $[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$ (unit direction of the axis), getting $[x, y, z] \bullet [\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}] = \frac{x+y+z}{\sqrt{3}}$. By Pythagoras, $d^2 = x^2 + y^2 + z^2 - \left[\frac{x+y+z}{\sqrt{3}}\right]^2$.

Answer:

$$\frac{M}{V}\int_0^1\int_0^{1-z}\int_0^{1-y-z}\left[x^2+y^2+z^2-\frac{(x+y+z)^2}{3}\right]dx\,dy\,dz =$$

$$4M\int_0^1\int_0^{1-z}\int_0^{1-y-z}\left[x^2+y^2+z^2-xy-xz-yz\right]dx\,dy\,dz$$

$$12M\int_0^1\int_0^{1-z}\int_0^{1-y-z}\left[x^2-xz\right]dx\,dy\,dz$$

[due to symmetry] $=$

$$12M\int_0^1\int_0^{1-z}\left[\frac{(1-y-z)^3}{3}-\frac{(1-y-z)^2}{2}z\right]dy\,dz =$$

$$M\int_0^1[(1-z)^4-2(1-z)^3z]\,dz =$$

$$M\left[-\frac{(1-z)^5}{5}+2\frac{(1-z)^4}{4}z+2\frac{(1-z)^5}{20}\right]_0^1 = \frac{M}{10}.$$

8. A container is made of a spherical shell of radius $1$ and height $h$. Find:

   (a) The shell's surface area.

   Solution: $\mathbf{r}(u,v)=[u,v,-\sqrt{1-u^2-v^2}] \Rightarrow$
   $\mathbf{r}_u=[1,0,\frac{u}{\sqrt{1-u^2-v^2}}]$ and $\mathbf{r}_v=[0,1,\frac{v}{\sqrt{1-u^2-v^2}}] \Rightarrow$

$$dA=\sqrt{\left(1+\frac{u^2}{1-u^2-v^2}\right)\left(1+\frac{v^2}{1-u^2-v^2}\right)-\frac{u^2v^2}{(1-u^2-v^2)^2}}\,du\,dv \equiv$$

$\sqrt{\frac{1}{1-u^2-v^2}}\, du\, dv$.

Answer: $\iint\limits_{u^2+v^2<h(2-h)} \frac{du\, dv}{\sqrt{1-u^2-v^2}} =$

$\int\limits_0^{2\pi} \int\limits_0^{\sqrt{h(2-h)}} \frac{r\, dr}{\sqrt{1-r^2}}\, d\varphi = 2\pi[-\sqrt{1-r^2}]_{r=0}^{\sqrt{h(2-h)}} =$

$2\pi[1-\sqrt{1-h(2-h)}] = 2\pi h$.

(b) The container's volume:

Solution: Since the $z$-thickness (depth) equals $\sqrt{1-x^2-y^2} - (1-h)$, all we need is $\iint\limits_{x^2+y^2<h(2-h)} \left[\sqrt{1-x^2-y^2} - (1-h)\right] dx\, dy =$

$\int\limits_0^{2\pi} \int\limits_0^{\sqrt{h(2-h)}} [\sqrt{1-r^2} - 1 + h] \cdot r\, dr\, d\varphi =$

$2\pi \left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}} - (1-h)\frac{r^2}{2}\right]_{r=0}^{\sqrt{h(2-h)}} =$

$2\pi[-\frac{1}{3}(1-h)^3 - (1-h)\frac{h(2-h)}{2} + \frac{1}{3}] =$

$\pi h^2 \left(1-\frac{h}{3}\right)$.

9. Evaluate $\oint\limits_{\mathcal{C}} [(x+y)\, dx + (2x-z)\, dy + (y+z)\, dz]$, where $\mathcal{C}$ is the closed curve

consisting of three straight-line segments connecting $[2, 0, 0]$ to $[0, 3, 0]$, that to $[0, 0, 6]$, and back to $[2, 0, 0]$.

Solution: Applying the Stokes' Theorem, which enables us to trade three line integrals (the three segments would require individual parametrization) for one surface integral, we first compute $Curl(\mathbf{g}) = [2, 0, 1]$, then $\mathbf{r}(u, v) = [u, v, 6 - 3u - 2v]$ (note that $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$ is the equation of the corresponding plane) $\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = [1, 0, -3] \times [0, 1, -2] = [3, 2, 1]$ which has the correct orientation.

Answer: $\iint\limits_{\substack{u>0 \\ v>0 \\ 3u+2v<6}} [2, 0, 1] \bullet [3, 2, 1] \, du \, dv = 7 \times Area = 7 \times \frac{2\times 3}{2} = 21$

[Verify by computing the line integral (broken onto three parts) directly].

10. Evaluate $\oint\limits_{\mathcal{C}} [yz \, dx + xz \, dy + xy \, dz]$, where $\mathcal{C}$ is the intersection of $x^2 + 9y^2 = 9$ and $z = 1 + y^2$ oriented counterclockwise when

viewed from above (in terms of $z$).

Solution: Applying the same Stokes' Theorem, we get $Curl(\mathbf{g}) \equiv [0,0,0]$.

Answer: $0$.

We will verify this by evaluating the line integral *directly*: $\mathbf{r}(t) = [3\cos t, \sin t, 1 + \sin^2 t]$ parametrizes the curve $(0 < t < 2\pi) \Rightarrow \int_0^{2\pi} [(1+\sin^2 t)\sin t,\ 3(1 + \sin^2 t)\cos t,\ 3\sin t\cos t] \bullet [-3\sin t,\ \cos t,\ 2\sin t\cos t]\, dt = \int_0^{2\pi} [-3\sin^2 t(1+\sin^2 t) + 3(1+\sin^2 t)(1 - \sin^2 t) + 6\sin^2 t(1 - \sin^2 t)]\, dt = \int_0^{2\pi} (3 + 3\sin^2 t - 12\sin^4 t)\, dt = 2\pi \times \left(3 + 3\times\frac{1}{2} - 12\times\frac{3}{8}\right) = 0$ [check].

Note that $\int_0^{2\pi} \sin^{2n} t\, dt = \int_0^{2\pi} \cos^{2n} t\, dt = 2\pi \times \frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\times\frac{7}{8}\times \ldots . \times \frac{2n-1}{2n}$.

11. In Physics we learned that the gravitational force of a 'solid' (i.e. 3-D) body exerted

on a point-like particle at $\mathbf{R} \equiv [X, Y, Z]$ is given by

$$\mu \iiint\limits_{\mathcal{V}} \rho(\mathbf{r}) \frac{\mathbf{r} - \mathbf{R}}{|\mathbf{r} - \mathbf{R}|^3} \, dV$$

where $\mu$ is a constant, $\rho$ is the body's mass density, and $\mathcal{V}$ is its 'volume' (i.e. 3-D extent). [Here we are integrating a vector field in the *componentwise* (scalar) sense, i.e. these are effectively *three* volume integrals, not one].

Prove that, when the body is *spherical* (of radius $a$) and $\rho$ is a function of $r$ *only* (placing the coordinate origin at the body's center), this force equals

$$\mu M \cdot \frac{-\mathbf{R}}{|\mathbf{R}|^3}$$

where $M$ is the body's total mass.

Solution: First we notice that $\frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3} \equiv \boldsymbol{\nabla}_{\mathbf{R}} \frac{1}{|\mathbf{r}-\mathbf{R}|}$, where $\boldsymbol{\nabla}_{\mathbf{R}} \equiv [\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}]$. This implies that $\mu \iiint\limits_{\mathcal{V}} \rho(\mathbf{r}) \frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3} \, dV \equiv$

$$\mu \boldsymbol{\nabla}_{\mathbf{R}} \iiint\limits_{\mathcal{V}} \rho(\mathbf{r}) \frac{1}{|\mathbf{r}-\mathbf{R}|} \, dV$$

leading to a lot easier integration (also, now we need *one*, not three integrals).

Evaluating $\iiint\limits_{\mathcal{V}} \rho(\mathbf{r}) \frac{1}{|\mathbf{r}-\mathbf{R}|} \, dV$ (the so called gravitational potential) in spherical coordinates yields $\int\limits_0^a \rho(r) \int\limits_0^\pi \int\limits_0^{2\pi} \frac{r^2 \sin\theta \cdot d\varphi \, d\theta \, dr}{\sqrt{r^2+R^2-2Rr\cos\theta}}$ [note that $|\mathbf{r}-\mathbf{R}| = \sqrt{x^2+y^2+(z-R)^2}$, where we have conveniently chosen the direction of $\mathbf{R}$ (instead of the usual $z$) to correspond to $\theta = 0$]. This further equals $\frac{2\pi}{R} \int\limits_0^a \rho(r) \cdot r \cdot \left[\sqrt{r^2+R^2-2Rr\cos\theta}\right]_{\theta=0}^{\pi} dr = \frac{2\pi}{R} \int\limits_0^a \rho(r) \cdot r \cdot [(R+r)-(R-r)] \, dr = \frac{4\pi}{R} \int\limits_0^a \rho(r) \, r^2 \, dr = \frac{M}{R}$.

This proves our assertion, as $\boldsymbol{\nabla}_{\mathbf{R}}\frac{1}{R} = \boldsymbol{\nabla}_{\mathbf{R}}\frac{1}{\sqrt{X^2+Y^2+Z^2}} = \left[\frac{-X}{(X^2+Y^2+Z^2)^{\frac{3}{2}}}, \frac{-Y}{(X^2+Y^2+Z^2)^{\frac{3}{2}}}, \frac{-Z}{(X^2+Y^2+Z^2)^{\frac{3}{2}}}\right] = \frac{-\mathbf{R}}{R^3}$.

Now try to prove the original statement directly (*bypassing* the potential), you should not find it too difficult.