MATH 2F05
(APPLIED ADVANCED CALCULUS)
LECTURE NOTES

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Contents

Chapter 1 PREREQUISITES

A few high-school formulas

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ &\vdots\end{aligned}$$

the coefficients follow from PASCAL'S TRIANGLE: $\begin{array}{c}1\\ 1\ 1\\ 1\ 2\ 1\\ 1\ 3\ 3\ 1\\ 1\ 4\ 6\ 4\ 1\\ 1\ 5\ 10\ 10\ 5\ 1\\ \ldots\ldots\end{array}$, the expansion is called BINOMIAL.

Also:

$$\begin{aligned}a^2-b^2 &= (a-b)\,(a+b)\\ a^3-b^3 &= (a-b)(a^2+ab+b^2)\\ &\vdots\end{aligned}$$

(do you know how to continue)?

Understand the **basic rules** of **algebra**: addition and multiplication, individually, are COMMUTATIVE and ASSOCIATIVE, when combined they follow the DISTRIBUTIVE law

$$(a+b+c)(d+e+f+h)=ad+ae+af+ah+bd+be+bf+bh+cd+ce+cf+ch$$

(each term from the left set of parentheses with each term on the right).

Polynomials

their DEGREE, the notion of individual COEFFICIENTS, basic operations including **synthetic division**, e.g.

$$\begin{array}{l}(x^3-3x^2+2x-4)\div(x-2) = x^2-x\ \ [\text{QUOTIENT}]\\ x^3-2x^2 \qquad\qquad \text{(subtract)}\\ \qquad -x^2+2x-4\\ \qquad -x^2+2x \qquad \text{(subtract)}\\ \qquad\qquad\qquad -4\ \ [\text{REMAINDER}]\end{array}$$

which implies that $(x^3-3x^2+2x-4)=(x^2-x)(x-2)-4$. The quotient's degree equals the degree of the dividend (the original polynomial) minus the degree of the divisor. The remainder's degree is always less than the degree of the divisor. When the remainder is zero, we have found two FACTORS of the dividend.

Exponentiation and logarithm

Rules of *exponentiation*:

$$\begin{aligned} a^A \cdot a^B &= a^{A+B} \\ (a^A)^B &= a^{AB} \end{aligned}$$

Also note that

$$(a^A)^B \neq a^{(A^B)}$$

The solution to

$$a^x = A$$

is:

$$x = \log_a(A)$$

(the *inverse function* to exponentiation). When $a = e$ $(= 2.7183...)$, this is written as

$$x = \ln(A)$$

and called **natural logarithm**.

Its basic **rules** are:

$$\begin{aligned} \ln(A \cdot B) &= \ln(A) + \ln(B) \\ \ln(A^B) &= B \cdot \ln(A) \\ \log_a(A) &= \frac{\ln(A)}{\ln(a)} \end{aligned}$$

Geometric series

First **infinite**:

$$1 + a + a^2 + a^3 + a^4 + ... = \frac{1}{1-a}$$

when $|a| < 1$ (understand the issue of series convergence)

and then **finite** (truncated):

$$1 + a + a^2 + a^3 + ... + a^N = \frac{1 - a^{N+1}}{1-a}$$

valid for all $a \neq 1$, (we don't need $a = 1$, why?).

Trigonometric formulas

such as, for example

$$(\sin a)^2 + (\cos a)^2 \equiv 1$$

and

$$\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned}$$

Our angles are always in RADIANS.

Inverse trigonometric functions: $\arcsin(x)$ – don't use the $\sin^{-1}(x)$ notation!

Solving equations

▶Single Unknown◀

Linear: $2x = 7$ (quite trivial).

Quadratic: $x^2 - 5x + 6 = 0 \quad \Rightarrow \quad x_{1,2} = \frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - 6} = \begin{cases} 3 \\ 2 \end{cases}$

This further implies that $x^2 - 5x + 6 = (x-3)(x-2)$.

Cubic and beyond: will be discussed when needed.

Can we solve **other** (non-polynomial) equations? Only when the left hand side involves a COMPOSITION of functions which we know how to invert individually, for example

$$\ln\left[\sin\left(\frac{1}{1+x^2}\right)\right] = -3$$

should pose no difficulty.

On the other hand, a simpler looking

$$\sin(x) = \frac{x}{2}$$

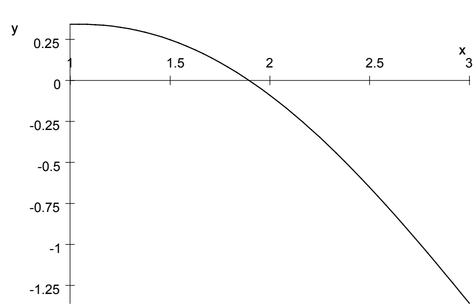
can be solved only *numerically*, usually by **Newton's technique**, which works as follows:

To solve an equation of the $f(x) = 0$ type, we start with an initial value x_0 which should be reasonably close to a root of the equation (found graphically), and then follow the *tangent* straight line from $[x_0, f(x_0)]$ till we cross the x-axis at x_1. This is repeated until the consecutive x-values no longer change.

In summary:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

To solve $\sin(x) - \frac{x}{2} = 0$ (our previous example) we first look at its graph



which indicates that there is a root close to $x = 1.9$. Choosing this as our x_0 we get: $x_1 = 1.9 - \frac{\sin(1.9)-0.95}{\cos(1.9)-0.5} = 1.895506$, $x_2 = x_1 - \dfrac{\sin(x_1) - \frac{x_1}{2}}{\cos(x_1) - \frac{1}{2}} = 1.895494$, after which the values no longer change. Thus $x = 1.895494$ is a solution (in this case, not the only one) to the original equation. ■

▶More Than One Unknown◀

We all know how to solve **linear sets** (systems) of equations: 2×2 (for sure, e.g. $\begin{matrix} 2x - 3y = 4 \\ 3x + y = -5 \end{matrix}$ $\Leftrightarrow$ [add 3×Eq.2 to Eq.1] $\begin{matrix} 11x = -11 \\ 3x + y = -5 \end{matrix}$ $\Rightarrow x = -1$ and $y = -2$), 3×3 (still rather routinely, I hope), 4×4 (gets tedious).

How about any **other** (nonlinear) case? We can try eliminating one unknown from one equation and substituting into the other equation, but this will work (in the 2×2 case) only if we are *very* lucky. We have to admit that we don't know how to solve most of these equations (and, in this course, we will have to live with that).

Differentiation

Interpretation: Slope of a tangent straight line.

Using **three** basic **rules** (PRODUCT, QUOTIENT and CHAIN) one can differentiate just about any function, repeatedly if necessary (i.e. finding the second, third, ... derivative), for example:

$$\frac{d}{dx}\left(x \sin x^2\right) = \sin x^2 + 2x^2 \cos x^2$$

[note that $\sin x^2 \equiv \sin(x^2)$ and *not* $(\sin x)^2$ – I will always be careful with my notation, using parentheses whenever an ambiguity may arise].

The main **formulas** are

$$\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$$

and

$$\frac{d}{dx}(e^{\beta x}) = \beta e^{\beta x}$$

The **product rule** can be extended to the second derivative:

$$(f \cdot g)'' = f'' \cdot g + 2f' \cdot g' + f \cdot g''$$

the third:

$$(f \cdot g)''' = f''' \cdot g + 3f'' \cdot g' + 3f' \cdot g'' + f \cdot g'''$$

etc. (Pascal's triangle again).

▶Partial Derivatives◀

Even when the function is BIVARIATE (of two variables), or MULTIVARIATE (of several variables), we always differentiate with respect to only one of the variables at a time (keeping the others constant). Thus, no new rules are needed, the only new thing is a more elaborate **notation**, e.g.

$$\frac{\partial^3 f(x,y)}{\partial x^2 \partial y}$$

(A function of a single variable is called UNIVARIATE).

▶Taylor (Maclaurin) Expansion◀

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + ...$$

alternate notation for $f^{iv}(0)$ is $f^{(4)}(0)$.

Remember the expansions of at least these functions:

$$\begin{aligned} e^x &= 1 + x + x^2/2 + x^3/3! + x^4/4! + ... \\ \sin(x) &= x - x^3/3! + x^5/5! - x^7/7! + ... \\ \ln(1+x) &= x - x^2/2 + x^3/3 - x^4/4 + ... \qquad \text{(no factorials)} \end{aligned}$$

and of course

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ...$$

The **bivariate** extension of Taylor's expansion (in a rather symbolic form):

$$\begin{aligned} f(x,y) &= f(0,0) + x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y} + \frac{1}{2}\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right\}^2 f(0,0) \\ &+ \frac{1}{3!}\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right\}^3 f(0,0) + ... \end{aligned}$$

where the partial derivatives are to be applied to $f(x,y)$ only. Thus

$$\left\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right\}^3 f \equiv x^3\frac{\partial^3 f}{\partial x^3} + 3x^2y\frac{\partial^3 f}{\partial x^2 \partial y} + 3xy^2\frac{\partial^3 f}{\partial x \partial y^2} + y^3\frac{\partial^3 f}{\partial y^3}$$

etc.

Basic limits

Rational expressions, e.g.

$$\lim_{n\to\infty} \frac{2n^2+3}{n^2-4n+1} = 2$$

Special limit:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

and also

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a \equiv \exp(a)$$

(introducing an alternate notation for e^a.

L'hôpital rule (to deal with the $\frac{0}{0}$ case):

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{(\sin x)'}{(x)'} = \frac{\cos(0)}{1} = 1$$

Integration

Interpretation: Area between x-axis and $f(x)$ (*up* is positive, *down* is negative).

Basic **formulas:**

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} \qquad \alpha \neq -1$$
$$\int e^{\beta x} dx = \frac{e^{\beta x}}{\beta}$$
$$\int \frac{dx}{x} = \ln|x|$$

etc. (use tables).

Useful **techniques and tricks**:

1. SUBSTITUTION (change of variable), for example
$$\int \frac{dx}{\sqrt{5x-2}} = \frac{2}{5}\int \frac{z\,dz}{z} = \frac{2}{5}z = \frac{2}{5}\sqrt{5x-2}$$
where $z = \sqrt{5x-2}$, thus $x = \dfrac{z^2+2}{5}$, and $\frac{dx}{dz} = \frac{2}{5}z \Rightarrow dx = \frac{2}{5}z\,dz$.

2. BY PARTS
$$\int fg'dx = fg - \int f'g\,dx$$
for example $\int xe^{-x}dx = \int x(-e^{-x})'dx = -xe^{-x} + \int e^{-x}dx = -(x+1)e^{-x}$.

3. PARTIAL FRACTIONS: When integrating a rational expression such as, for example $\dfrac{1}{(1+x)^2(1+x^2)}$, first rewrite it as
$$\frac{a}{1+x} + \frac{b}{(1+x)^2} + \frac{c+dx}{1+x^2}$$
then solve for a, b, c, d based on
$$1 = a(1+x)(1+x^2) + b(1+x^2) + (c+dx)(1+x)^2 \tag{\#1}$$
This can be done most efficiently by substituting $x = -1 \Rightarrow 1 = 2b \Rightarrow b = \frac{1}{2}$. When this ($b = \frac{1}{2}$) is substituted back into (#1), $(1+x)$ can be factored out, resulting in
$$\frac{1-x}{2} = a(1+x^2) + (c+dx)(1+x) \tag{\#2}$$
Then substitute $x = -1$ again, get $a = \frac{1}{2}$, and further simplify (#2) by factoring out yet another $(1+x)$:
$$-\frac{x}{2} = c + dx$$
yielding the values of $c = 0$ and $d = -\frac{1}{2}$. In the new form, the function can be integrated easily. ■

Basic **properties**:

$$\int [cf(x) + dg(x)]dx = c\int f(x)\,dx + d\int g(x)\,dx \qquad \text{(linear)}$$

and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \text{(try picture it)}$$

These properties are shared with (actually inherited from) SUMMATION; we all know that, for example

$$\sum_{i=1}^{\infty}\left(3a^i - \frac{1}{i^2}\right) = 3\sum_{i=1}^{\infty} a^i - \sum_{i=1}^{\infty}\frac{1}{i^2}$$

Have some basic ideas about **double** (and, eventually, **triple**) integration, e.g.

$$\iint\limits_R (x^2y - 3xy^3)dA$$

where R is a REGION OF INTEGRATION (it may be a square, a triangle, a circle and such), and dA a symbol for an infinitesimal area (usually visualized as a small rectangle within this region).

The simplest case is the so called SEPARABLE integral, where the function (of x and y) to be integrated is a *product* of a function of x (only) times a function of y (only), and the region of integration is a GENERALIZED RECTANGLE (meaning that the limits of integration don't depend on each other, but any of them may be infinite).

For example:

$$\int\limits_{y=0}^{\infty}\int\limits_{x=1}^{3} \sin(x)e^{-y}dA = \int\limits_{x=1}^{3}\sin(x)dx \times \int\limits_{y=0}^{\infty} e^{-y}dy$$

which is a product of two ordinary integrals.

In **general**, a double integral must be converted to two consecutive *univariate* integrations, the first in x and the second in y (or vice versa – the results must be identical). Notationally, the x and y limits, and the dx and dy symbols should follow proper 'nesting' (similar to two sets of parentheses), since that is effectively the logic of performing the integration in general.

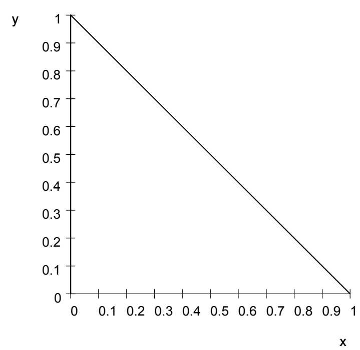
More complicated cases may require a **change of variable**s. This may lead to POLAR COORDINATES and requires understanding of the JACOBIAN (details to be discussed when needed).

Geometry

▶In Two Dimensions◀

Know that $y = 1 - x$ is an equation of a **straight line**, be able to identify its slope and intercept.

The collection of points which lie below this straight line and are in the first QUADRANT defines the following two-dimensional **region:**



There are **two** other **ways** of describing it:

1. $0 \le y \le 1 - x$ (CONDITIONAL y-range), where $0 \le x \le 1$ (MARGINAL x-range) – visualize this as the triangle being filled with *vertical lines*, the marginal range describing the x-scale 'shadow' of the region.

2. $0 \le x \le 1 - y$ (conditional x-range), where $0 \le y \le 1$ (marginal y-range) – *horizontal lines.*

The two descriptions will usually *not* be this *symmetric*; try doing the same thing with $y = 1 - x^2$ (a branch of a PARABOLA). For the 'horizontal-line' description, you will get: $0 \le x \le \sqrt{1 - y^2}$, where $0 \le y \le 1$.

Regions of this kind are frequently encountered in two-dimensional integrals. The vertical (or horizontal)-line description facilitates **constructing** the inner and outer **limits of** the corresponding (consecutive) **integration**. Note that in this context we don't have to worry about the boundary points being included (e.g. $0 \le x \le 1$) or excluded ($0 < x < 1$), this makes no difference to the integral's value.

Recognize **equation** of a CIRCLE (e.g. $x^2 + y^2 - 3x + 4y = 9$) be able to identify its center and radius; also that of PARABOLA and HYPERBOLA.

▶In Three Dimensions◀

Understand the standard (RECTANGULAR) coordinate system and how to use it to display points, lines and planes.

Both (fixed) points and (free) vectors have three COMPONENTS, written as: $[2, -1, 4]$ (sometimes organized in a column). Understand the difference between the two: POINT describes a single *location*, VECTOR defines a *length* (or strength, in physics), *direction* and *orientation.*

Would you be able to **describe** a *three-dimensional* **region** using x, y and z (consecutive) ranges, similar to (but more complicated than) the 2-D case?

Later on we discuss vector *functions* (physicists call them 'fields'), which assign to each point in space a vector. We also learn how to deal with CURVES and SURFACES.

Matrices

such as, for example $\begin{bmatrix} 4 & -6 & 0 \\ -3 & 2 & 1 \\ 9 & -5 & -3 \end{bmatrix}$ which is a 3×3 ($n \times m$ in general) array of numbers (these are called the matrix ELEMENTS).

Understand the following **definitions**:

SQUARE matrix ($n = m$), UNIT matrix ($\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$), ZERO matrix;

and **operations**:

matrix TRANSPOSE ($\begin{bmatrix} 4 & -3 & 9 \\ -6 & 2 & -5 \\ 0 & 1 & -3 \end{bmatrix}$), matrix ADDITION (for same-size matrices), MULTIPLICATION ($\begin{bmatrix} 3 & -2 & 0 \\ 5 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 6 & -9 \end{bmatrix}$, $[n \times m][m \times k] = [n \times k]$ in terms of DIMENSIONS), INVERSE $\begin{bmatrix} 4 & -6 & 0 \\ -3 & 2 & 1 \\ 9 & -5 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{9}{2} & \frac{3}{2} \\ 0 & 3 & 1 \\ \frac{3}{4} & \frac{17}{2} & \frac{5}{2} \end{bmatrix}$ (later, we review its construction), and DETERMINANT $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

For two square matrices, their **product** is not necessarily COMMUTATIVE, e.g.

$$\begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ -5 & -23 \end{bmatrix}, \quad \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -9 & 20 \\ 8 & -8 \end{bmatrix}.$$

Notation:

$\mathbb{I}$ stands for the *unit* matrix, $\mathbb{O}$ for the *zero* matrix, $\mathbb{A}^T$ for *transpose*, $\mathbb{A}^{-1}$ for *inverse*, $|\mathbb{A}|$ for the *determinant*, $\mathbb{AB}$ for *multiplication* (careful with the order); A_{23} is the second-row, third-column *element* of $\mathbb{A}$.

A few basic **rules**:

$\mathbb{AI} = \mathbb{IA} = \mathbb{A}$, $\mathbb{AO} = \mathbb{OA} = \mathbb{O}$, $\mathbb{AA}^{-1} = \mathbb{A}^{-1}\mathbb{A} = \mathbb{I}$, $(\mathbb{AB})^T = \mathbb{B}^T\mathbb{A}^T$, and $(\mathbb{AB})^{-1} = \mathbb{B}^{-1}\mathbb{A}^{-1}$, whenever the dimensions allow the operation.

Let us formally prove the second last equality:

$$\left\{(\mathbb{AB})^T\right\}_{ij} = (\sum_k A_{ik}B_{kj})^T = \sum_k A_{jk}B_{ki} \overset{\substack{\text{Why can we} \\ \text{interchange?}}}{=} \sum_k B_{ki}A_{jk} = \sum_k \{\mathbb{B}^T\}_{ik}\{\mathbb{A}^T\}_{kj} =$$
$\{\mathbb{B}^T\mathbb{A}^T\}_{ij}$ for each i and j. $\square$

Complex numbers

Understand the **basic algebra** of *adding* and *subtracting* (trivial), *multiplying* $(3-2i)(4+i) = 12+2-8i+3i = 14-5i$ (distributive law plus $i^2=-1$) and *dividing*:

$$\frac{3-2i}{4+i} = \frac{(3-2i)(4-i)}{(4+i)(4-i)} = \frac{10-11i}{17} = \frac{10}{17} - \frac{11}{17}i$$

Notation:

Complex CONJUGATE of a number $z = x + yi$ is $\overline{z} = x - yi$ (change the sign of i).

MAGNITUDE (absolute value): $|z| = \sqrt{x^2+y^2}$.

REAL and IMAGINARY parts are $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$, respectively.

Polar representation

of $z = x + yi = r\, e^{i\theta} = r\cos(\theta) + i\, r\sin(\theta)$, where $r = |z|$ and $\theta = \arctan_2(y,x)$ which is called the ARGUMENT of z and is chosen so that $\theta \in [0, 2\pi)$.

Here, we have used

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

which follows from Maclaurin expansion (try the proof).

When multiplying two complex numbers in polar representation, their magnitudes multiply as well, but their 'arguments' only *add*. When raising a complex number to an integer n, the resulting argument (angle) is simply $n\theta$.

This is quite useful when taking a large integer power of a complex number, for example: $(3+2i)^{31} = (9+4)^{\frac{31}{2}} \left[\cos\left(31\arctan(\tfrac{2}{3})\right) + i\sin\left(31\arctan(\tfrac{2}{3})\right)\right] = 1.5005\times10^{17} - 1.0745\times10^{17} i$ (give at least *four significant digits* in all your answers).

Part I

ORDINARY DIFFERENTIAL EQUATIONS

General review of differential equations

►(Single) Ordinary Differential Equation◄

There is only *one independent* variable, usually called x, and *one dependent* variable (a function of x), usually called $y(x)$. The equation involves x, $y(x)$, $y'(x)$, and possibly higher derivatives of $y(x)$, for example: $y''(x) = \dfrac{1 + y(x) \cdot y'(x)}{1 + x^2}$. We usually leave out the argument of y to simplify the notation, thus:

$$y'' = \frac{1 + yy'}{1 + x^2}$$

The highest derivative is the ORDER of the equation (this one is of the *second* order).

Our task is to find the (unknown) function $y(x)$ which meets the equation. The solution is normally a FAMILY of functions, with as many extra parameters as the order of the equation (these are usually called C_1, C_2, C_3,...).

In the next chapter, we study *first-order ODE*. Then, we move on to higher-order ODE, but, for these, we restrict our attention almost entirely to the LINEAR [in y and its derivatives, e.g. $y'' + \sin x \cdot y' - (1 + x^2) \cdot y = e^{-x}$] case with CONSTANT COEFFICIENTS [example: $y'' - 2y' + 3y = e^{2x}$]. When the right hand side of such an equation is zero, the equation is called HOMOGENOUS.

►A Set (or System) of ODEs◄

Here, we have *several dependent* (unknown) functions y_1, y_2, y_3, ... of still a *single independent* variable x (sometimes called t). We will study only a special case of these systems, when the equations (and we need as many equations as there are unknown functions) are of the FIRST ORDER, LINEAR in y_1, y_2, y_3, ... and its derivative, and having CONSTANT COEFFICIENTS.

Example:

$$\begin{aligned} \dot{y}_1 &= 2y_1 - 3y_2 + t^2 \\ \dot{y}_2 &= y_1 + 5y_2 + \sin t \end{aligned}$$

(when the independent variable is t, $\dot{y}_1$, $\dot{y}_2$, ... is a common notation for the first derivatives).

►Partial Differential Equations (PDE)◄

They usually have a *single dependent* variable and *several independent* variables. The derivatives are then automatically of the partial type. It is unlikely that we will have time to discuss even a brief introduction to these. But you will study them in Physics, where you also encounter *systems* of PDEs (*several dependent and independent* variables), to be solved by all sorts of ingenious techniques. They are normally tackled on an individual basis only.

Chapter 2 FIRST-ORDER DIFFERENTIAL EQUATIONS

The most **general form** of such an equation (assuming we can solve it for y', which is usually the case) is

$$y' = f(x, y)$$

where $f(x, y)$ is any expression involving both x and y. We know how to solve this equation (analytically) in only a few special cases (to be discussed shortly).

Graphically, the situation is easier: using an x–y set of coordinates, we can draw (at as many points as possible) the slope $\equiv f(x, y)$ of a solution passing through the corresponding (x, y) point and then, by attempting to connect these, visualize the family of all solutions. But this is not very accurate nor practical.

Usually, there is a whole **family** of solutions which covers the whole x–y plane, by curves which don't intersect. This means that exactly one solution passes through each point. Or, equivalently: given an extra condition imposed on the solution, namely $y(x_0) = y_0$ where x_0 and y_0 are two specific numbers (the so called INITIAL CONDITION), this singles out a *unique* solution. But sometimes it happens that no solution can be found for some initial conditions, and more than one (even infinitely many) solutions for others.

We will look at some of these issues in more detail later on, let us now go over the *special cases* of the first-order ODE which we know how to solve:

'Trivial' equation

This case is so simple to solve that it is usually not even mentioned as such, but we want to be systematic. The equation has the **form**:

$$y' = f(x)$$

i.e. y' is expressed as a function of x *only*.

It is quite obvious that the **general solution** is

$$y(x) = \int f(x)dx + C$$

(graphically, it is the same curve slid vertically up and down). Note that even in this simplest case we cannot always find an analytical solution (we don't know how to integrate *all* functions).

EXAMPLE: $y' = \sin(x)$.

Solution: $y(x) = -\cos(x) + C$.

Separable equation

Its general **form** is:

$$y' = h(x) \cdot g(y)$$

(a product of a function of x times a function of y).

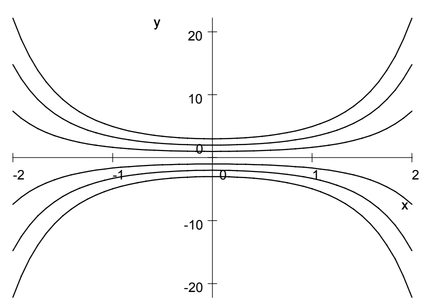
Solution: Writing y' as $\frac{dy}{dx}[= h(x) \cdot g(y)]$ we can achieve the actual separation (of x from y), thus:

$$\frac{dy}{g(y)} = h(x)dx$$

where the left and the right hand sides can be *individually* integrated (in terms of y and x, respectively), a constant C added (to the right hand side only, why is this sufficient?), and the resulting equation solved for y whenever possible (to get the so called EXPLICIT solution). If the equation cannot be solved for y, we leave it in the so called IMPLICIT form.

EXAMPLES:

1. $y' = x \cdot y \Rightarrow \frac{dy}{y} = x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + \tilde{C} \Rightarrow y = \pm e^{\tilde{C}} \cdot e^{\frac{x^2}{2}} \equiv Ce^{\frac{x^2}{2}}$ (by definition of a new C, which may be positive or negative). Let us plot these with $C = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$ to visualize the whole family of solutions:



(essentially the same curve, expanded or compressed along the y-direction).

2. $9yy' + 4x = 0 \Rightarrow 9y\,dy = -4x\,dx \Rightarrow 9\frac{y^2}{2} = -4\frac{x^2}{2} + \tilde{C} \Rightarrow y^2 + \frac{4}{9}x^2 = C$ (by intelligently redefining the constant). We will leave the answer in its explicit form, which clearly shows that the family of solutions are ellipses centered on the origin, with the vertical versus horizontal diameter in the $2:3$ ratio.

3. $y' = -2xy \Rightarrow \frac{dy}{y} = -2x\,dx \Rightarrow \ln|y| = -x^2 + \tilde{C} \Rightarrow y = Ce^{-x^2}$ (analogous to Example 1).

4. $(1 + x^2)y' + 1 + y^2 = 0$, with $y(0) = 1$ (initial value problem).

   Solution: $\frac{dy}{1+y^2} = -\frac{dx}{1+x^2} \Rightarrow \arctan(y) = -\arctan(x) + \tilde{C} \equiv \arctan(C) - \arctan(x) \Rightarrow \quad y = \tan(\arctan(C) - \arctan(x)) = \frac{C-x}{1+Cx}$ [recall the $\tan(\alpha - \beta)$ formula]. To find C we solve $1 = \frac{C-0}{1+C\times 0} \Rightarrow C = 1$.

   Answer: $y(x) = \frac{1-x}{1+x}$.

   Check: $(1 + x^2)\frac{d}{dx}\left(\frac{1-x}{1+x}\right) + 1 + \left(\frac{1-x}{1+x}\right)^2 = 0$ ✓.

Scale-independent equation

looks as follows:

$$y' = g\left(\frac{y}{x}\right)$$

(note that the right hand side does not change when replacing $x \to ax$ and, simultaneously, $y \to ay$, since a will cancel out).

Solve by introducing a *new dependent* variable $u(x) = \dfrac{y(x)}{x}$ or, equivalently, $y(x) = x \cdot u(x)$. This implies $y' = u + xu'$; substituted into the original equation yields:

$$xu' = g(u) - u$$

which is *separable* in x and u:

$$\frac{du}{g(u) - u} = \frac{dx}{x}$$

Solve the separable equation for $u(x)$, and convert to $y(x) = xu(x)$.

EXAMPLES:

1. $2xyy' - y^2 + x^2 = 0 \Rightarrow y' = \dfrac{y/x}{2} - \dfrac{1}{2y/x} \Rightarrow xu' = -\dfrac{u^2+1}{2u} \Rightarrow \dfrac{2u\,du}{u^2+1} = -\dfrac{dx}{x} \Rightarrow \ln(1+u^2) = -\ln|x| + \tilde{C} \Rightarrow u^2 + 1 = \dfrac{2C}{x}$ (factor of 2 is introduced for future convenience) $\Rightarrow y^2 + x^2 - 2Cx = 0 \Rightarrow y^2 + (x - C)^2 = C^2$ (let us leave it in the implicit form). This is a family of circles having a center at any point of the x-axis, and being tangent to the y-axis

2. $x^2y' = y^2 + xy + x^2 \Rightarrow y' = \left(\dfrac{y}{x}\right)^2 + \dfrac{y}{x} + 1 \Rightarrow xu' = u^2 + 1 \Rightarrow \dfrac{du}{1+u^2} = \dfrac{dx}{x} \Rightarrow \arctan(u) = \ln|x| + C \Rightarrow u = \tan(\ln|x| + C) \Rightarrow y = x \cdot \tan(\ln|x| + C)$ ■

▶Modified Scale-Independent◀

$$y' = \frac{y}{x} + g\left(\frac{y}{x}\right) \cdot h(x)$$

The same substitution gives

$$xu' = g(u) \cdot h(x)$$

which is also *separable*.

The main point is to be able to *recognize* that the equation is of this type.

EXAMPLE:

$y' = \dfrac{y}{x} + \dfrac{2x^3\cos(x^2)}{y} \Rightarrow xu' = \dfrac{2x^2\cos(x^2)}{u} \Rightarrow u\,du = 2x\cos(x^2)\,dx \Rightarrow \dfrac{u^2}{2} = \sin(x^2) + \tilde{C} \Rightarrow u = \pm\sqrt{2\sin(x^2) + C} \Rightarrow y = \pm x\sqrt{2\sin(x^2) + C}$ ■

▶Any Other Smart Substitution◀

(usually suggested), which makes the equation separable.

EXAMPLES:

1. $(2x-4y+5)y'+x-2y+3=0$ [suggestion: introduce: $v(x)=x-2y(x)$, i.e. $y=\dfrac{x-v}{2}$ and $y'=\dfrac{1-v'}{2}$]$\Rightarrow$ $(2v+5)\dfrac{1-v'}{2}+v+3=0 \Rightarrow -(v+\frac{5}{2})v'+2v+\frac{11}{2}=0 \Rightarrow \dfrac{v+\frac{5}{2}}{v+\frac{11}{4}}\,dv=2\,dx \Rightarrow \left(1-\dfrac{\frac{1}{4}}{v+\frac{11}{4}}\right)dv=2\,dx \Rightarrow v-\frac{1}{4}\ln|v+\frac{11}{4}|=2x+C \Rightarrow x-2y-\frac{1}{4}\ln|x-2y+\frac{11}{4}|=2x+C$. We have to leave the solution in the implicit form because we cannot solve for y, except numerically – it would be a painstaking procedure to draw even a simple graph now).

2. $y'\cos(y)+x\sin(y)=2x$, seems to suggest $\sin(y)\equiv v(x)$ as the new dependent variable, since $v'=y'\cos(y)$ [by chain rule]. The new equation is thus simply: $v'+xv=2x$, which is linear (see the next section), and can be solved as such: $\frac{dv}{v}=-x\,dx \Rightarrow \ln|v|=-\frac{x^2}{2}+\tilde{c} \Rightarrow v=ce^{-\frac{x^2}{2}}$, substitute: $c'e^{-\frac{x^2}{2}}-xce^{-\frac{x^2}{2}}+xce^{-\frac{x^2}{2}}=2x \Rightarrow c'=2xe^{\frac{x^2}{2}} \Rightarrow c(x)=2e^{\frac{x^2}{2}}+C \Rightarrow v(x)=2+Ce^{-\frac{x^2}{2}} \Rightarrow y(x)=\arcsin\left(2+Ce^{-\frac{x^2}{2}}\right)$ ■

Linear equation

has the **form** of:

$$y'+g(x)\cdot y=r(x)$$

[both $g(x)$ and $r(x)$ are arbitrary – but specific – functions of x].

The solution is constructed in two stages, by the so called

▶Variation-of-Parameters Technique◀

which works as follows:

1. Solve the *homogeneous* equation $y'=-g(x)\cdot y$, which is *separable*, thus:

$$y_h(x)=c\cdot e^{-\int g(x)dx}$$

2. Assume that c itself is a *function* of x, substitute $c(x)\cdot e^{-\int g(x)dx}$ back into the full equation, and solve the resulting [trivial] differential equation for $c(x)$.

EXAMPLES:

1. $y'+\dfrac{y}{x}=\dfrac{\sin x}{x}$

   Solve $y'+\dfrac{y}{x}=0 \Rightarrow \dfrac{dy}{y}=-\dfrac{dx}{x} \Rightarrow \ln|y|=-\ln|x|+\tilde{c} \Rightarrow y=\dfrac{c}{x}$.

   Now substitute this to the original equation: $\dfrac{c'}{x}-\dfrac{c}{x^2}+\dfrac{c}{x^2}=\dfrac{\sin x}{x} \Rightarrow c'=\sin x \Rightarrow c(x)=-\cos x+C$ (the big C being a true constant) $\Rightarrow y(x)=$

$-\dfrac{\cos x}{x} + \dfrac{C}{x}$. The solution has always the form of $y_p(x) + Cy_h(x)$, where $y_p(x)$ is a PARTICULAR solution to the full equation, and $y_h(x)$ solves the homogeneous equation only.

Let us verify the former: $\dfrac{d}{dx}\left(-\dfrac{\cos x}{x}\right) - \dfrac{\cos x}{x^2} = \dfrac{\sin x}{x}$ (check).

2. $y' - y = e^{2x}$.

   First $y' - y = 0 \Rightarrow \frac{dy}{y} = dx \Rightarrow y = ce^x$.

   Substitute: $c'e^x + ce^x - ce^x = e^{2x} \Rightarrow c' = e^x \Rightarrow c(x) = e^x + C \Rightarrow y(x) = e^{2x} + Ce^x$.

3. $xy' + y + 4 = 0$

   Homogeneous: $\frac{dy}{y} = -\frac{dx}{x} \Rightarrow \ln|y| = -\ln|x| + c \Rightarrow y = \dfrac{c}{x}$

   Substitute: $c' - \frac{c}{x} + \frac{c}{x} = -4 \Rightarrow c(x) = -4x + C \Rightarrow y(x) = -4 + \dfrac{C}{x}$.

4. $y' + y \cdot \tan(x) = \sin(2x)$, $y(0) = 1$.

   Homogeneous: $\frac{dy}{y} = \frac{-\sin x\, dx}{\cos x} \Rightarrow \ln|y| = \ln|\cos x| + \tilde{c} \Rightarrow y = c \cdot \cos x$.

   Substitute: $c' \cos x - c \sin x + c \sin x = 2 \sin x \cos x \Rightarrow c' = 2 \sin x \Rightarrow c(x) = -2 \cos x + C \Rightarrow y(x) = -2 \cos^2 x + C \cos x$ [$\cos^2 x$ is the usual 'shorthand' for $(\cos x)^2$].

   To find the value of C, solve: $1 = -2 + C \Rightarrow C = 3$.

   The final answer is thus: $y(x) = -2\cos^2 x + 3 \cos x$.

   To verify: $\frac{d}{dx}[-2\cos^2 x + 3\cos x] + [-2\cos^2 x + 3\cos x] \cdot \frac{\sin x}{\cos x} = 2 \cos x \sin x$ (check).

5. $x^2 y' + 2xy - x + 1 = 0$, $y(1) = 0$

   Homogeneous [realize that here $-x + 1$ is the non-homogeneous part]: $\frac{dy}{y} = -2\frac{dx}{x} \Rightarrow \ln|y| = -2\ln|x| + \tilde{C} \Rightarrow y = \frac{c}{x^2}$

   Substitute: $c' - \dfrac{2c}{x^3} + \dfrac{2c}{x^3} - x + 1 = 0 \Rightarrow c' = x - 1 \Rightarrow c = \frac{x^2}{2} - x + C \Rightarrow y = \dfrac{1}{2} - \dfrac{1}{x} + \dfrac{C}{x^2}$

   To meet the initial-value condition: $0 = \frac{1}{2} - 1 + C \Rightarrow C = \frac{1}{2}$

   Final answer: $y = \dfrac{(1-x)^2}{2x^2}$.

   Verify: $x^2 \dfrac{d}{dx}\left(\dfrac{(1-x)^2}{2x^2}\right) + 2x\left(\dfrac{(1-x)^2}{2x^2}\right) - x + 1 \equiv 0$ ✓.

6. $y' - \frac{2y}{x} = x^2 \cos(3x)$

   First: $\frac{dy}{y} = 2\frac{dx}{x} \Rightarrow \ln|y| = 2\ln|x| + \tilde{c} \Rightarrow y = cx^2$

Substitute: $c'x^2+2cx-2cx = x^2\cos(3x) \Rightarrow c' = \cos(3x) \Rightarrow c = \dfrac{\sin(3x)}{3}+C \Rightarrow$ $y = \frac{x^2}{3}\sin(3x) + Cx^2$

To verify the particular solution: $\frac{d}{dx}\left(\frac{x^2}{3}\sin(3x)\right) - \frac{2x}{3}\sin(3x) = x^2\cos(3x)$ ✓

Bernoulli equation

$$y' + f(x)\cdot y = r(x)\cdot y^a$$

where a is a specific (constant) exponent.

Introducing a *new dependent* variable $u = y^{1-a}$, i.e. $y = u^{\frac{1}{1-a}}$, one gets: $\frac{1}{1-a}u^{\frac{1}{1-a}-1}u'$ [chain rule] $+f(x)\cdot u^{\frac{1}{1-a}} = r(x)\cdot u^{\frac{a}{1-a}}$.

Multiplying by $(1-a)u^{-\frac{a}{1-a}}$ results in:

$$u' + (1-a)f(x)\cdot u = (1-a)r(x)$$

which is *linear* in u' and u (i.e., of the previous type), and solved as such.

The answer is then easily converted back to $y = u^{\frac{1}{1-a}}$.

EXAMPLES:

1. $y' + xy = \frac{x}{y}$ (Bernoulli, $a=-1$, $f(x)\equiv x$, $g(x)\equiv x$) $\Rightarrow u' + 2xu = 2x$ where $y = u^{\frac{1}{2}}$

   Solving as linear: $\frac{du}{u} = -2x\,dx \Rightarrow \ln|u| = -x^2 + \tilde{c} \Rightarrow u = c\cdot e^{-x^2}$

   Substitute: $c'e^{-x^2} - 2xce^{-x^2} + 2xce^{-x^2} = 2x \Rightarrow c' = 2xe^{x^2} \Rightarrow c(x) = e^{x^2}+C \Rightarrow$ $u(x) = 1 + Ce^{-x^2} \Rightarrow y(x) = \pm\sqrt{1+Ce^{-x^2}}$ (one can easily check that this is a solution with either the + or the − sign).

2. $2xy' = 10x^3y^5 + y$ (terms reshuffled a bit). Bernoulli with $a=5$, $f(x) = -\frac{1}{2x}$, and $g(x) = 5x^2$

   This implies $u' + \frac{2}{x}u = -20x^2$ with $y = u^{-\frac{1}{4}}$

   Solving as linear: $\frac{du}{u} = -2\frac{dx}{x} \Rightarrow \ln|u| = -2\ln|x| + \tilde{c} \Rightarrow u = \frac{c}{x^2}$

   Substituted back into the full equation: $\frac{c'}{x^2} - 2\frac{c}{x^3} + 2\frac{c}{x^3} = -20x^2 \Rightarrow c' = -20x^4 \Rightarrow c(x) = -4x^5+C \Rightarrow u(x) = -4x^3+\dfrac{C}{x^2} \Rightarrow y(x) = \pm\left(-4x^3+\dfrac{C}{x^2}\right)^{-\frac{1}{4}}$.

3. $2xyy' + (x-1)y^2 = x^2e^x$, Bernoulli with $a=-1$, $f(x) = \frac{x-1}{2x}$, and $g(x) = \frac{x}{2}e^x$

   This translates to: $u' + \frac{x-1}{x}u = xe^x$ with $y = u^{\frac{1}{2}}$

   Solving homogeneous part: $\frac{du}{u} = (\frac{1}{x} - 1)\,dx \Rightarrow \ln|u| = \ln|x| - x + \tilde{c} \Rightarrow$ $u = cxe^{-x}$

   Substituted: $c'xe^{-x} + ce^{-x} - cxe^{-x} + (x-1)ce^{-x} = xe^x \Rightarrow c' = e^{2x} \Rightarrow$ $c(x) = \frac{1}{2}e^{2x} + C \Rightarrow u(x) = \frac{x}{2}e^x + Cxe^{-x} \Rightarrow y(x) = \pm\sqrt{\frac{x}{2}e^x + Cxe^{-x}}$ ■

Exact equation

First we have to explain the **general idea** behind this type of equation:

Suppose we have a function of x and y, $f(x,y)$ say. Then $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is the so called TOTAL DIFFERENTIAL of this function, corresponding to the function's increase when its arguments change from (x,y) to $(x+dx, y+dy)$. By setting this quantity equal to zero, we are effectively demanding that the function does not change its value, i.e. $f(x,y) = C$ (constant). The last equation is then an [implicit] solution to

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

(the corresponding *exact* equation).

EXAMPLE: Suppose $f(x,y) = x^2y - 2x$. This means that $(2xy-2)\,dx + x^2\,dy = 0$ has a simple solution $x^2y - 2x = C \Rightarrow y = \frac{2}{x} + \frac{C}{x^2}$. Note that the differential equation can be also re-written as: $y' = 2\frac{1-xy}{x^2}$, $x^2y' + 2xy = 2$, etc. [coincidentally, this equation is also linear; we can thus double-check the answer].

We must now try to *reverse* the process, since in the actual situation we will be given the *differential equation* and asked to *find* the corresponding $f(x,y)$.

There are then **two issues** to be settled:

1. How do we verify that the equation *is* exact?
2. Knowing it is, how do we solve it?

To answer the first question, we recall that $\frac{\partial^2 f}{\partial x\,\partial y} \equiv \frac{\partial^2 f}{\partial y\,\partial x}$. Thus, $g(x,y)\,dx + h(x,y)\,dy = 0$ is exact **if and only if**

$$\frac{\partial g}{\partial y} \equiv \frac{\partial h}{\partial x}$$

As to **solving** the equation, we proceed in three stages:

1. Find $G(x,y) = \int g(x,y)\,dx$ (considering y a constant).
2. Construct $H(y) = h(x,y) - \frac{\partial G}{\partial y}$ [must be a function of y only, as $\frac{\partial H}{\partial x} = \frac{\partial h}{\partial x} - \frac{\partial^2 G}{\partial x\,\partial y} = \frac{\partial g}{\partial y} - \frac{\partial g}{\partial y} \equiv 0$].
3. $f(x,y) = G(x,y) + \int H(y)\,dy$
   [Proof: $\frac{\partial f}{\partial x} = \frac{\partial G}{\partial x} = g$ and $\frac{\partial f}{\partial y} = \frac{\partial G}{\partial y} + H = h$ $\square$].

Even though this *looks complicated*, one must realize that the individual steps are rather trivial, and exact equations are therefore *easy* to solve.

EXAMPLE: $2x\sin(3y)\,dx + (3x^2\cos(3y) + 2y)\,dy = 0$.

Let us first verify that the equation *is* exact: $\frac{\partial}{\partial y} 2x\sin(3y) = 6x\cos 3y$, $\frac{\partial}{\partial x}(3x^2\cos(3y) + 2y) = 6x\cos 3y$ (check)

Solving it: $G = x^2\sin(3y)$, $H = 3x^2\cos(3y) + 2y - 3x^2\cos(3y) = 2y$, $f(x, y) = x^2\sin(3y) + y^2$.

Answer: $y^2 + x^2\sin(3y) = C$ (implicit form) ■

▶Integrating Factors◀

Any first-order ODE (e.g. $y' = \frac{y}{x}$) can be expanded in a form which makes it *look like* an exact equation, thus: $\frac{dy}{dx} = \frac{y}{x} \Rightarrow y\,dx - x\,dy = 0$. But since $\frac{\partial(y)}{\partial y} \neq -\frac{\partial(x)}{\partial x}$, this equation is *not* exact.

The good news is that there is always a function of x and y, say $F(x, y)$, which can multiply any such equation (a legitimate modification) to make it exact. This function is called an **integrating factor**.

The bad news is that there is no general procedure for finding $F(x, y)$ [if there were, we would know how to solve *all* first-order differential equations – too good to be true].

Yet, there are **two special cases** when it *is* possible:
Let us write the differential equation in its 'look-like-exact' form of

$$P(x, y)dx + Q(x, y)dy = 0$$

where $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ (thus the equation is *not* exact yet). One can find an integrating factor from

1. $$\frac{d\ln F}{dx} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}$$

   iff the right hand side of this equation is a function of x *only*

   Proof: $FPdx + FQdy = 0$ is exact when $\frac{\partial(FP)}{\partial y} = \frac{\partial(FQ)}{\partial x} \Rightarrow F\frac{\partial P}{\partial y} = \frac{dF}{dx} \cdot Q + F\frac{\partial Q}{\partial x}$ *assuming* that F is a function of x *only*. Solving for $\frac{\frac{dF}{dx}}{F}$ results in $\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}$. When the last expression contains no y, we simply integrate it (with respect to x) to find $\ln F$. Otherwise (when y does *not* cancel out of the expression), the formula is meaningless. □

2. or from

   $$\frac{d\ln F}{dy} = \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P}$$

   iff the right hand side is a function of y *only*.

EXAMPLES:

1. Let us try solving our $y\,dx - x\,dy = 0$. Since $\dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = -\dfrac{2}{x}$, we have $\ln F = -2\int \frac{dx}{x} = -2\ln x$ (no need to bother with a constant) $\Rightarrow F = \frac{1}{x^2}$. Thus $\frac{y}{x^2}\,dx - \frac{1}{x}\,dy = 0$ must be exact (check it). Solving it gives $-\frac{y}{x} = \tilde{C}$, or $y = Cx$.

   The original equation is, coincidentally, also separable (sometimes it happens that an equation can be solved in more than one way), so we can easily verify that the answer is correct.

   But this is not the end of this example yet! We can also get: $\dfrac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} = -\dfrac{2}{y}$, which implies that $\ln F = -2\int \frac{dy}{y} = -2\ln y \Rightarrow F = \frac{1}{y^2}$. Is this also an integrating factor?

   The answer is yes, there are infinitely many of these; one can multiply an integrating factor (such as $\frac{1}{x^2}$) by any function of what we know must be a constant ($-\frac{y}{x}$ in our case, i.e. the left hand side of our solution). Since $\dfrac{1}{y^2} = \dfrac{1}{x^2} \cdot \frac{1}{\left(-\frac{y}{x}\right)^2}$, this is also an integrating factor of our equation. One can verify that, using this second integration factor, one still obtains the same simple $y = Cx$ solution.

   To formalize our **observation**: When $g\,dx + h\,dy = 0$ is exact (i.e. $g = \dfrac{\partial f}{\partial x}$ and $h = \frac{\partial f}{\partial y}$), so is $R(f)\,g\,dx + R(f)\,h\,dy = 0$ where R is any function of f.

   Proof: $\frac{\partial (Rg)}{\partial y} = \frac{dR}{df} \cdot \frac{\partial f}{\partial y} \cdot g + R \cdot \frac{\partial g}{\partial y} = \frac{dR}{df} \cdot h \cdot g + R \cdot \frac{\partial g}{\partial y}$. Similarly $\frac{\partial (Rh)}{\partial x} = \frac{dR}{df} \cdot \frac{\partial f}{\partial x} \cdot h + R \cdot \frac{\partial h}{\partial x} = \frac{dR}{df} \cdot g \cdot h + R \cdot \frac{\partial h}{\partial x}$. Since $\frac{\partial g}{\partial y} \equiv \frac{\partial h}{\partial x}$, the two expressions are identical. □

2. $(2\cos y + 4x^2)\ dx = x\sin y\ dy$ [i.e. $Q = -x\sin y$].

   Since $\dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{-2\sin y + \sin y}{-x\sin y} = \frac{1}{x}$ we get $\ln F = \int \frac{1}{x}\,dx = \ln x \Rightarrow F = x$.

   $$\left(2x\cos y + 4x^3\right)\ dx - x^2\sin y\,dy = 0$$

   is therefore exact, and can be solved as such: $x^2\cos y + x^4 = C \Rightarrow y = \arccos\left(\dfrac{C}{x^2} - x^2\right)$.

3. $(3xe^y + 2y)\,dx + (x^2e^y + x)\,dy = 0$.

   Trying again $\dfrac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \dfrac{xe^y + 1}{x^2e^y + x} = \dfrac{1}{x}$, which means that $\ln F = \int \frac{dx}{x} = \ln x \Rightarrow F = x$.

   $$(3x^2e^y + 2xy)\,dx + (x^3e^y + x^2)\,dy = 0$$

   is exact. Solving it gives: $x^3e^y + x^2y = C$ [implicit form]. ■

More 'exotic' equations

and methods of solving them:

There are many other types of first-order ODEs which can be solved by all sorts of ingenious techniques (stressing that our list of 'solvable' equations has been far from complete). We will mention only one, for illustration:

Clairaut equation:

$$y = xy' + g(y')$$

where g is an arbitrary function. The idea is to introduce $p(x) \equiv y'(x)$ as an unknown function, differentiate the original equation with respect to x, obtaining $p = p + xp' + p'g'(p) \Rightarrow p' \cdot (x + g'(p)) = 0$. This implies that either $p \equiv y' = C \Rightarrow$

$$y = xC + g(C)$$

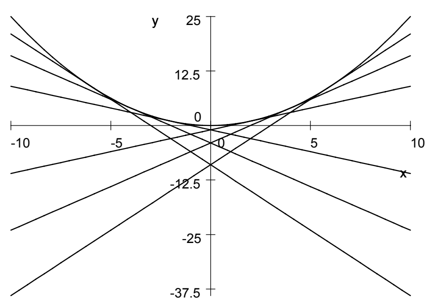
which represents a family of REGULAR solutions (all straight lines), or

$$x = -g'(p)$$

which, when solved for p and substituted back into $y = xp + g(p)$ provides the so called SINGULAR solution (an ENVELOPE of the regular family).

EXAMPLE:

$(y')^2 - xy' + y = 0$ (terms reshuffled a bit) is solved by either $y = Cx - C^2$, or $x = 2p \Rightarrow p = \frac{x}{2} \Rightarrow y = xp - p^2 = \frac{x^2}{4}$ (singular solution). Let us display them graphically:



Note that for an initial condition below or at the parabola two possible solutions exist, above the parabola there is none.

This concludes our discussion of Clairaut equation.

And finally, a **useful trick** worth mentioning: When the (general) equation appears more complicated in terms of y rather than x [e.g. $(2x + y^4)y' = y$], one can try reversing the role of x and y (i.e. considering x as the *dependent* variable and y as the *independent* one). All it takes is to replace $y' \equiv \frac{dy}{dx}$ by $\frac{1}{\frac{dx}{dy}}$, for example (using the previous equation): $\frac{dx}{dy} = 2\frac{x}{y} + y^3$ (after some simplification). The last equation is linear (in x and $\frac{dx}{dy}$) and can be solved as such:

$\frac{dx}{x} = 2\frac{dy}{y} \Rightarrow \ln|x| = 2\ln|y| + \tilde{c} \Rightarrow x(y) = y^2 \cdot c(y)$.

Substituted into the full equation: $2yc + y^2\frac{dc}{dy} = 2\frac{y^2 c}{y} + y^3 \Rightarrow \frac{dc}{dy} = y \Rightarrow c(y) = \frac{y^2}{2} - C$ [the minus sign is more convenient here] $\Rightarrow x = \frac{y^4}{2} - Cy^2$.

This can now be solved for y in terms of x, to get a solution to the original equation: $y = \pm\sqrt{C \pm \sqrt{C^2 + 2x}}$.

Applications

▶Of Geometric Kind◀

1. Find a curve such that (from each of its points) the distance to the origin is the same as the distance to the intersection of its NORMAL (i.e. perpendicular straight line) with the x-axis.

   Solution: Suppose $y(x)$ is the equation of the curve (yet unknown). The equation of the normal is

   $$Y - y = -\frac{1}{y'} \cdot (X - x)$$

   where (x, y) [fixed] are the points of the curve, and (X, Y) [variable] are the points of the normal [which is a straight line passing through (x, y), with its slope equal minus the reciprocal of the curve's slope y']. This normal intersects the x-axis at $Y = 0$ and $X = yy' + x$. The distance between this and the original (x, y) is $\sqrt{(yy')^2 + y^2}$, the distance from (x, y) to $(0, 0)$ is $\sqrt{x^2 + y^2}$. These two distances are equal when $y^2(y')^2 = x^2$, or $y' = \pm\frac{x}{y}$. This is a separable differential equation easy to solve: $y^2 \pm x^2 = C$. The curves are either circles centered on $(0, 0)$ [yes, that checks, right?], or hyperbolas [with $y = \pm x$ as special cases].

2. Find a curve whose normals (all) pass through the origin.

   Solution (we can guess the answer, but let us do it properly): Into the same equation of the curve's normal (see above), we substitute 0 for both X and Y, since the straight line must pass through $(0, 0)$. This gives: $-y = \frac{x}{y'}$, which is simple to solve: -$y\,dy = x\,dx \Rightarrow x^2 + y^2 = C$ (circles centered on the origin – we knew that!).

3. A family of curves covering the whole x–y plane enables one to draw lines perpendicular to these curves. The collection of all such lines is yet another family of curves ORTHOGONAL (i.e. perpendicular) to the original family. If we can find the differential equation $y' = f(x, y)$ having the original family of curves as its solution, we can find the corresponding orthogonal family by solving $y' = -\frac{1}{f(x,y)}$. The next set of examples relates to this.

   (a) The original family is described by $x^2 + (y - C)^2 = C^2$ with C arbitrary (i.e. collection of circles tangent to the x-axis at the origin). To find the corresponding differential equation, we differentiate the original equation with respect to x: $2x + 2(y - C)y' = 0$, solve for $y' = \frac{x}{C-y}$, and then eliminate C by solving the original equation for C, thus: $x^2 + y^2 - 2Cy = 0 \Rightarrow C = \frac{x^2+y^2}{2y}$, further implying $y' = \dfrac{x}{\frac{x^2+y^2}{2y} - y} = \frac{2xy}{x^2-y^2}$. To find the orthogonal family, we solve $y' = \frac{y^2-x^2}{2xy}$ [scale-independent equation solved earlier]. The answer is: $(x - C)^2 + y^2 = C^2$, i.e. collection of circles tangent to the y-axis at the origin.

(b) Let the original family be circles centered on the origin (it should be clear what the orthogonal family is, but again, let's solve it anyhow): $x^2+y^2 = C^2$ describes the original family, $2x + 2yy' = 0$ is the corresponding differential equation (equivalent to $y' = -\frac{x}{y}$, this time there is no C to eliminate). The orthogonal family is the solution to $y' = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x} \Rightarrow$ $y = Cx$ (all straight lines passing through the origin).

(c) Let the original family be described by $y^2 = x + C$ (the $y^2 = x$ parabola slid horizontally). The corresponding differential equation is $2yy' = 1$, the 'orthogonal' equation: $y' = -2y$.
Answer: $\ln|y| = -2x + \tilde{C}$. or $y = Ce^{-2x}$ (try to visualize the curves).

(d) Finally, let us start with $y = Cx^2$ (all parabolas tangent to the x-axis at the origin). Differentiating: $y' = 2Cx \Rightarrow$ [since $C = \frac{y}{x^2}$] $y' = 2\frac{y}{x}$. The 'orthogonal' equation is $y' = -\frac{x}{2y} \Rightarrow y^2 + \frac{x^2}{2} = C$ [collection of ellipses centered on the origin, with the x-diameter being $\sqrt{2}$ times bigger than the y-diameter).

4. The position of four ships on the ocean is such that the ships form vertices of a square of length L. At the same instant each ship fires a missile that directs its motion towards the missile on its right. Assuming that the four missiles fly horizontally and with the same constant speed, find the path of each.

   Solution: Let us place the origin at the center of the original square. It should be obvious that when we find one of the four paths, the other three can be obtained just by rotating it by 90, 180 and 270 degrees. This is actually true for the missiles' positions at any instant of time. Thus, if a missile is at (x, y), the one to its right is at $(y, -x)$ [(x, y) rotated by 90^o]. If $y(x)$ is the resulting path for the first missile, $Y - y = y' \cdot (X - x)$ is the straight line of its immediate direction. This straight line must pass through $(y, -x)$ [that's where the other missile is, at the moment]. This means that, when we substitute y and $-x$ for X and Y, respectively, the equation must hold: $-x - y = y' \cdot (y - x)$. And this is the differential equation to solve (as scale independent): $xu' + u(= y' = \frac{x+y}{x-y}) = \frac{1+u}{1-u} \Rightarrow xu' = \frac{1+u^2}{1-u} \Rightarrow \frac{1-u}{1+u^2}du = \frac{dx}{x} \Rightarrow \arctan(u) - \frac{1}{2}\ln(1+u^2) = \ln|x| + \tilde{C} \Rightarrow \dfrac{e^{\arctan(u)}}{\sqrt{1+u^2}} = Cx$. This solution becomes a lot easier to understand in polar coordinates [$\theta = \arctan(\frac{y}{x})$, and $r = \sqrt{x^2+y^2}$], where it looks like this: $r = \dfrac{e^{\theta}}{C}$ (a spiral).

▶To Physics◀

If a hole is made at a bottom of a container, water will flow out at the rate of $a\sqrt{h}$, where a is established based on the size (and to some extent, the shape) of the opening, but to us it is simply a constant, and h is the height of the (remaining) water, which varies in time. Time t is the independent variable. Find $h(t)$ as a function of t for:

1. A cylindrical container of radius r.

   Solution: First we have to establish the volume V of the remaining water as a function of height. In this case we get simply $V(h(t)) = \pi r^2 h(t)$. Differentiating with respect to t we get: $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$. This in turn must be equal to $-a\sqrt{h(t)}$ since the rate at which the water is flowing out must be equal to the rate at which its volume is *decreasing*. Thus $\pi r^2 \dot{h} = -a\sqrt{h}$, where $\dot{h} \equiv \frac{dh}{dt}$. This is a simple (separable) differential equation for $h(t)$, which we solve by $\dfrac{dh}{\sqrt{h}} = -\dfrac{a}{\pi r^2} dt \Rightarrow \dfrac{h^{1/2}}{\frac{1}{2}} = -\dfrac{at}{\pi r^2} + C \Rightarrow \sqrt{h} = -\dfrac{at}{2\pi r^2} + \sqrt{h_0}$ [h_0 is the initial height at time $t = 0$], or equivalently $t = \frac{2\pi r^2}{a}\left(\sqrt{h_0} - \sqrt{h}\right)$.

   Subsidiary: What percentage of time is spent emptying the last 20% of the container? Solution: $t_1 = \frac{2\pi r^2}{a}\sqrt{h_0}$ is the time to fully empty the container (this follows form our previous solution with $h = 0$). $t_{0.8} = \frac{2\pi r^2}{a}\left(\sqrt{h_0} - \sqrt{\frac{h_0}{5}}\right)$ is the time it takes to empty the first 80% of the container. The answer: $\dfrac{t_1 - t_{0.8}}{t_1} = \sqrt{\frac{1}{5}} = 44.72\%$.

2. A conical container with the top radius (at h_0) equal to r.

   Solution: $V(h(t)) = \frac{1}{3}\pi h \left(r\dfrac{h}{h_0}\right)^2$ [follows from $\int\limits_0^h \mu \left(\frac{rx}{h_0}\right)^2 dx$]. Note that one fifth of the full *volume* corresponds to $h = \left(\frac{1}{5}\right)^{1/3} h_0$ (i.e. 58.48% of the full *height*!), obtained by solving $\frac{1}{3}\pi\left(r\dfrac{h}{h_0}\right)^2 h = \frac{1}{3}\pi r^2 \dfrac{h_0}{5}$ for h. Thus $\dfrac{2\pi r^2}{3h_0^2} h^2 \dot{h} = -a\sqrt{h}$ is the (separable) equation to solve, as follows: $h^{3/2}\, dh = -\dfrac{3ah_0^2}{2\pi r^2}\, dt \Rightarrow h^{5/2} = -\dfrac{15ah_0^2}{4\pi r^2} t + h_0^{5/2} \Leftrightarrow t = \dfrac{4\pi r^2}{15ah_0^2}\left(h_0^{5/2} - h^{5/2}\right)$. This implies $t_1 = \dfrac{4\pi r^2 \sqrt{h_0}}{15a}$ and $t_{0.8} = \dfrac{4\pi r^2 \sqrt{h_0}}{15a} \cdot \left[1 - \left(\frac{1}{5}\right)^{5/6}\right] \Rightarrow \dfrac{t_1 - t_{0.8}}{t_1} = \left(\frac{1}{5}\right)^{5/6} = 26.15\%$.

3. A hemisphere of radius R (this is the radius of its top rim, also equal to the water's full height).

   Solution: $V(h(t)) = \frac{1}{3}\pi h^2(3R - h)$ [follows from $\int\limits_0^h \pi\left[R^2 - (R-x)^2\right] dx$]. Making the right hand side equal to $\frac{2}{3}\pi R^3/5$ and solving for h gives the height of the 20% (remaining) volume. This amounts to solving $z^3 - 3z^2 + \frac{2}{5} = 0$ (a cubic equation) for $z \equiv \frac{h}{R}$. We will discuss formulas for solving cubic and quartic equations in the next chapter, for the time being we extract the desired root by Newton's technique: $z_{n+1} = z_n - \dfrac{z_n^3 - 3z_n^2 + \frac{2}{5}}{3z_n^2 - 6z_n}$ starting with say $z_0 = 0.3 \Rightarrow z_1 = 0.402614 \Rightarrow z_2 = 0.391713 \Rightarrow z_3 = z_4 = ... = 0.391600 \Rightarrow h = 0.391600R$. Finish as your assignment.

Chapter 3 SECOND ORDER DIFFERENTIAL EQUATIONS

These are substantially more difficult to solve than first-order ODEs, we will thus concentrate mainly on the simplest case of *linear* equations with *constant coefficients*. Only in the following introductory section we look at two special **non-linear** cases:

Reducible to first order

If, in a second-order equation

► y is missing ◄

(does not appear explicitly; only x, y' and y'' do), then $y' \equiv z(x)$ can be considered the unknown function of the equation. In terms of $z(x)$, the equation is of the first order only, and can be solved as such. Once we have the explicit expression for $z(x)$, we need to integrate it with respect to x to get $y(x)$.

The final solution will thus have two arbitrary constants, say C_1 and C_2 – this is the case of all second-order equations, in general. With two arbitrary constants we need **two conditions** to single out a unique solution. These are usually of two distinct types

1. INITIAL CONDITIONS: $y(x_0) = a$ and $y'(x_0) = b$ [x_0 is quite often 0], specifying a value and a slope of the function at a single point,

   or

2. BOUNDARY CONDITIONS: $y(x_1) = a$ and $y(x_2) = b$, specifying a value each at two distinct points.

EXAMPLES:

1. $y'' = y' \Rightarrow z' = z$ [separable] $\Rightarrow \dfrac{dz}{z} = dx \Rightarrow \ln|z| = x + \widetilde{C}_1 \Rightarrow z = C_1 e^x \Rightarrow y = C_1 e^x + C_2$. Let's impose the following initial conditions: $y(0) = 0$ and $y'(0) = 1$. By substituting into the general solution we get: $C_1 + C_2 = 0$ and $C_1 = 1 \Rightarrow C_0 = -1 \Rightarrow y = e^x - 1$ as the final answer.

2. $xy'' + y' = 0 \Rightarrow xz' + z = 0$ [separable] $\Rightarrow \dfrac{dz}{z} = -\dfrac{dx}{x} \Rightarrow \ln|z| = \ln|x| + \widetilde{C}_1 \Rightarrow z = \dfrac{C_1}{x} \Rightarrow y = C_1 \ln|x| + C_2$. Let us make this into a boundary-value problem: $y(1) = 1$ and $y(3) = 0 \Rightarrow C_2 = 1$ and $C_1 \ln 3 + C_2 = 0 \Rightarrow C_1 = -\dfrac{1}{\ln 3} \Rightarrow y = 1 - \dfrac{\ln|x|}{\ln 3}$.

3. $xy'' + 2y' = 0 \Rightarrow xz' + 2z = 0$ [still separable] $\Rightarrow \dfrac{dz}{z} = -2\dfrac{dx}{x} \Rightarrow z = \dfrac{\widetilde{C}_1}{x^2} \Rightarrow y = \dfrac{C_1}{x} + C_2$. Sometimes the two extra conditions can be of a more bizarre

type: $y(2) = \frac{1}{2}$ *and* requiring that the solution intersects the $y = x$ straight line at the right angle. Translated into our notation: $y'(x_0) = -1$ where x_0 is a solution to $y(x) = x$, i.e. $-\dfrac{C_1}{x_0^2} = -1$ with $\dfrac{C_1}{x_0} + C_2 = x_0$. Adding the original $\dfrac{C_1}{2} + C_2 = \frac{1}{2}$, we can solve for $C_2 = 0$, $C_1 = 1$ and $x_0 = 1$ (that is where our solution intersects $y = x$). The final answer: $y(x) = \dfrac{1}{x}$. ■

The second type (of a second-order equation reducible to first order) has

▶ x missing ◀

(not appearing explicitly) [such as $y \cdot y'' + (y')^2 = 0$]. We again introduce $z \equiv y'$ as a new dependent variable, but this time we see it as a function of y, which becomes the *independent* variable of the new equation! Furthermore, since $y'' = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$ [chain rule], we replace y'' in the original equation by $\frac{dz}{dy} \cdot z$. We solve the resulting first-order equation for z (as a function of y), replace z by y' [thus creating another first-order equation, this time for $y(x)$] and solve again.

EXAMPLES:

1. $y \cdot y'' + (y')^2 = 0 \Rightarrow y\frac{dz}{dy}z + z^2 = 0$ [separable] $\Rightarrow \frac{dz}{z} = -\frac{dy}{y} \Rightarrow \ln|z| = -\ln|y| + \widetilde{C}_1 \Rightarrow z = \frac{\hat{C}_1}{y} \Rightarrow y' = \frac{\hat{C}_1}{y}$ [separable again] $\Rightarrow y\,dy = \hat{C}_1 \Rightarrow y^2 = C_1 x + C_2$.

2. $y'' + e^{2y}(y')^3 = 0 \Rightarrow \frac{dz}{dy}z + e^{2y}z^3 = 0 \Rightarrow \frac{dz}{z^2} = -e^{2y}dy \Rightarrow -\frac{1}{z} = -\frac{1}{2}e^{2y} - C_1 \Rightarrow z = \dfrac{1}{C_1 + \frac{1}{2}e^{2y}} \Rightarrow (C_1 + \frac{1}{2}e^{2y})dy = dx \Rightarrow C_1 y + \frac{1}{4}e^{2y} = x + C_2$.

3. $y'' + (1 + \frac{1}{y})(y')^2 = 0 \Rightarrow \frac{dz}{dy}z + (1 + \frac{1}{y})z^2 = 0 \Rightarrow \frac{dz}{z} = -(1 + \frac{1}{y})dy \Rightarrow \ln|z| = -\ln|y| - y + \widetilde{C}_1 \Rightarrow z = \frac{C_1}{y}e^{-y} \Rightarrow ye^y dy = C_1 dx \Rightarrow (y-1)e^y = C_1 x + C_2$. ■

Linear equation

The most general form is

$$y'' + f(x)y' + g(x)y = r(x) \qquad (*)$$

where f, g and r are *specific* functions of x. When $r \equiv 0$ the equation is called HOMOGENEOUS.

There is no general technique for solving this equation, but some **results** relating to it are worth quoting:

1. The general solution must look like this: $y = C_1 y_1 + C_2 y_2 + y_p$, where y_1 and y_2 are linearly independent 'BASIC' solutions (if we only knew how to find them!) of the corresponding *homogeneous* equation, and y_p is any PARTICULAR solution to the *full* equation. None of these are unique (e.g. $y_1 + y_2$ and $y_1 - y_2$ is yet another basic set, etc.).

2. When one basic solution (say y_1) of the *homogeneous* version of the equation is known, the other can be found by a technique called VARIATION OF PARAMETERS (V of P): Assume that its solution has the form of $c(x)y_1(x)$, substitute this *trial solution* into the equation and get a *first-order* differential equation for $c' \equiv z$.

EXAMPLES:

1. $y'' - 4xy' + (4x^2 - 2)y = 0$ given that $y_1 = \exp(x^2)$ is a solution (verify!). Substituting $y_T(x) = c(x) \cdot \exp(x^2)$ back into the equation [remember: $y_T' = c'y_1 + \{cy_1'\}$ and $y_T'' = c''y_1 + 2c'y_1' + \{cy_1''\}$; also remember that the c-proportional terms, i.e. $\{...\}$ *and* $(4x^2 - 2)cy_1$ must *cancel* out] yields $c'' \exp(x^2) + 4xc' \exp(x^2) - 4xc' \exp(x^2) = 0 \Rightarrow c'' = 0$. With $z \equiv c'$, the resulting equation is *always* of the first order: $z' = 0 \Rightarrow z = C_1 \Rightarrow c(x) = C_1 x + C_2$. Substituting back to y_T results in the general solution of the original homogeneous equation: $y = C_1 x \exp(x^2) + C_2 \exp(x^2)$. We can thus identify $x \exp(x^2)$ as our y_2.

2. $y'' + \frac{2}{x}y' + y = 0$ given that $y_1 = \frac{\sin x}{x}$ is a solution (verify!). $y_T = c(x)\frac{\sin x}{x}$ substituted: $c''\frac{\sin x}{x} + 2c'(\frac{\cos x}{x} - \frac{\sin x}{x^2}) + \frac{2}{x}c'\frac{\sin x}{x} = 0 \Rightarrow c'' \sin x + 2c' \cos x = 0 \Rightarrow$ $z' \sin x + 2z \cos x = 0 \Rightarrow \frac{dz}{z} = -2\frac{\cos x\, dx}{\sin x} \Rightarrow \ln|z| = -2\ln|\sin x| + \widetilde{C}_1 \Rightarrow$ $z = \frac{-C_1}{\sin^2 x} \Rightarrow c(x) = C_1\frac{\cos x}{\sin x} + C_2 \Rightarrow y = C_1\frac{\cos x}{x} + C_2\frac{\sin x}{x}$ [y_2 equals to $\frac{\cos x}{x}$]. ■

When both basic solutions of the homogeneous version are known, a *particular* solution to the full **non-homogeneous** equations can be found by using an extended V-of-P idea. This time we have two unknown 'parameters' called $u(x)$ and $v(x)$ [in the previous case there was only one, called $c(x)$]. Note that, in this context, 'parameters' are actually functions of x.

We now derive **general formulas** for $u(x)$ and $v(x)$:

The objective is to solve Eq. (*) assuming that y_1 and y_2 (basic solutions of the *homogeneous* version) are known. We then need to find y_p only, which we take to have a *trial* form of $u(x) \cdot y_1 + v(x) \cdot y_2$, with $u(x)$ and $v(x)$ yet to be found (the variable parameters). Substituting this into the full equation [note that terms proportional to $u(x)$, and those proportional to $v(x)$, cancel out], we get:

$$u''y_1 + 2u'y_1' + v''y_2 + 2v'y_2' + f(x)\,(u'y_1 + v'y_2) = r(x) \qquad \text{(V of P)}$$

This is a single differential equation for *two* unknown functions (u and v), which means we are free to impose yet another *arbitrary* constraint on u and v. This is chosen to simplify the previous equation, thus:

$$u'y_1 + v'y_2 = 0$$

which further implies (after one differentiation) that $u''y_1 + u'y_1' + v''y_2 + v'y_2' = 0$. The original (V of P) equation therefore simplifies to

$$u'y_1' + v'y_2' = r(x)$$

The last two (centered) equations can be solved (*algebraically*, using Cramer's rule) for

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

and

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

where the denominator it called the WRONSKIAN of the two basic solutions (these are linearly independent iff their Wronskian is nonzero; one can use this as a check – useful later when dealing with more than two basic solutions). From the last two expressions one can easily find u and v by an extra integration (the right hand sides are known functions of x).

EXAMPLE:

$y''-4xy'+(4x^2-2)y = 4x^4-3$. We have already solved the homogeneous version getting $y_1 = \exp(x^2)$ and $y_2 = x\exp(x^2)$. Using the previous two formulas we get $u' = \frac{\begin{vmatrix} 0 & x\exp(x^2) \\ 4x^4-3 & (1+2x^2)\exp(x^2) \end{vmatrix}}{\begin{vmatrix} \exp(x^2) & x\exp(x^2) \\ 2x\exp(x^2) & (1+2x^2)\exp(x^2) \end{vmatrix}} = (3-4x^4)x\exp(-x^2) \Rightarrow$ $u(x) = (\frac{5}{2}+4x^2+2x^4)e^{(-x^2)}+C_1$ and $v' = \frac{\begin{vmatrix} \exp(x^2) & 0 \\ 2x\exp(x^2) & 4x^4-3 \end{vmatrix}}{\begin{vmatrix} \exp(x^2) & x\exp(x^2) \\ 2x\exp(x^2) & (1+2x^2)\exp(x^2) \end{vmatrix}} =$ $(4x^4-3)e^{(-x^2)} \Rightarrow v(x) = -(3+2x^2)xe^{(-x^2)} + C_2$ [the last integration is a bit more tricky, but the result checks]. Simplifying $uy_1 + vy_2$ yields $(\frac{5}{2}+x^2) + C_1\exp(x^2) + C_2 x\exp(x^2)$, which identifies $\frac{5}{2}+x^2$ as a particular solution of the full equation (this can be verified easily). ■

Given three specific functions y_1, y_2 and y_p, it is possible to **construct** a differential **equation** of type (*) which has $C_1y_1 + C_2y_2 + y_p$ as its general solution (that's how I set up exam questions).

EXAMPLE:

Knowing that $y = C_1x^2 + C_2\ln x + \frac{1}{x}$, we first substitute x^2 and $\ln x$ for y in $y'' + f(x)y' + g(x)y = 0$ [the homogeneous version] to get:

$$2 + 2x \cdot f + x^2 \cdot g = 0$$
$$-\frac{1}{x^2} + \frac{1}{x} \cdot f + \ln x \cdot g = 0$$

and solve, *algebraically*, for $\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 2x & x^2 \\ \frac{1}{x} & \ln x \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ \frac{1}{x^2} \end{bmatrix} \Rightarrow f = \frac{-2\ln x - 1}{x(2\ln x - 1)}$ and $g = \frac{4}{x^2(2\ln x-1)}$. The left hand side of the equation is therefore $y'' +$

$\frac{-2\ln x-1}{x\,(2\ln x-1)}\,y'+\frac{4}{x^2\,(2\ln x-1)}\,y$ [one could multiply the whole equation by $x^2(2\ln x-1)$ to simplify the answer]. To ensure that $\frac{1}{x}$ is a particular solution, we substitute it into the left hand side of the last equation(for y), yielding $r(x)$ [$=\frac{3\,(2\ln x+1)}{x^3\,(2\ln x-1)}$ in our case]. The final answer is thus:

$$x^2(2\ln x-1)y''-x(2\ln x+1)y'+4y=\frac{3}{x}(2\ln x+1)$$

With constant coefficients

From now on we will assume that the two 'coefficients' $f(x)$ and $g(x)$ are x-independent constants, and call them a and b (to differentiate between the two cases). The equation we want to solve is then

$$y''+ay'+by=r(x)$$

with a and b being two specific *numbers*. We will start with the

►Homogeneous Case◄ $[r(x)\equiv 0]$

All we have to do is to find two linearly *independent* basic solutions y_1 and y_2, and then combine them in the $c_1y_1+c_2y_2$ manner (as we already know from the general case).

To achieve this, we *try* a solution of the following form:

$$y_T=e^{\lambda x}$$

where λ is a number whose value is yet to be determined. Substituting this into $y''+ay'+by=0$ and dividing by $e^{\lambda x}$ results in

$$\lambda^2+a\lambda+b=0$$

which is the so called **characteristic polynomial** for λ.

When this (quadratic) equation has two real roots the problem is solved (we have gotten our two basic solutions). What do we do when the two roots are complex, or when only a single root exists? Let us look at these possibilities, one by one.

1. **Two (distinct) real roots**.

   EXAMPLE: $y''+y'-2y=0\Rightarrow\lambda^2+\lambda-2=0$ [characteristic polynomial] $\Rightarrow\lambda_{1,2}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+2}=-\frac{1}{2}\pm\frac{3}{2}=1$ and -2. This implies $y_1=e^x$ and $y_2=e^{-2x}$, which means that the general solution is $y=C_1e^x+C_2e^{-2x}$.

2. **Two complex conjugate roots** $\lambda_{1,2}=p\pm i\,q$.

   This implies that $\tilde{y}_1=e^{px}[\cos(qx)+i\sin(qx)]$ and $\tilde{y}_2=e^{px}[\cos(qx)-i\sin(qx)]$ [remember that $e^{iA}=\cos A+i\sin A$]. But at this point we are interested in real solutions *only*, and these are complex. But we can take the following linear combination of the above functions: $y_1\equiv\frac{\tilde{y}_1+\tilde{y}_2}{2}=e^{px}\cos(qx)$ and

$y_2 \equiv \frac{\tilde{y}_1 - \tilde{y}_2}{2i} = e^{px}\sin(qx)$, and have a new, *equivalent*, basis set. The new functions are both *real*, thus the general solution can be written as

$$y = e^{px}[C_1 \cos(qx) + C_2 \sin(qx)]$$

One can easily verify that both y_1 and y_2 do (individually) meet the original equation.

EXAMPLE: $y'' - 2y' + 10y = 0 \Rightarrow \lambda_{1,2} = 1 \pm \sqrt{1-10} = 1 \pm 3i$. Thus $y = e^x[C_1 \cos(3x) + C_2 \sin(3x)]$ is the general solution.

3. **One (double) real root**.

   This can happen only when the original equation has the form of: $y'' + ay' + \frac{a^2}{4}y = 0$ (i.e. $b = \frac{a^2}{4}$). Solving for λ, one gets: $\lambda_{1,2} = -\frac{a}{2} \pm 0$ (double root). This gives us only one basic solution, namely $y_1 = e^{-\frac{a}{2}x}$; we can find the other by the V-of-P technique. Let us substitute the following trial solution $c(x) \cdot e^{-\frac{a}{2}x}$ into the equation getting (after we divide by $e^{-\frac{a}{2}x}$): $c'' - ac' + ac' = 0 \Rightarrow c'' = 0 \Rightarrow c(x) = C_1 x + C_2$. The trial solution thus becomes: $C_1 x e^{-\frac{a}{2}x} + C_2 e^{-\frac{a}{2}x}$, which clearly identifies

$$y_2 = x e^{-\frac{a}{2}x}$$

   as the second basic solution.

   Remember: For duplicate roots, the second solution can be obtained by multiplying the first basic solution by x.

   EXAMPLE: $y'' + 8y' + 16y = 0 \Rightarrow \lambda_{1,2} = -4$ (both). The general solution is thus $y = e^{-4x}(C_1 + C_2 x)$. Let's try finishing this as an initial-value problem [lest we forget]: $y(0) = 1$, $y'(0) = -3$. This implies $C_1 = 1$ and $-4C_1 + C_2 = -3 \Rightarrow C_2 = 1$. The final answer: $y = (1+x)e^{-4x}$. ■

For a second-order equation, these three possibilities cover the whole story.

▶Non-homogeneous Case◀

When any such equation has a nonzero right hand side $r(x)$, there are two possible ways of building a particular solution y_p:

▷ Using the **variation-of-parameters** formulas derived earlier for the general case.

EXAMPLES:

1. $y'' + y = \tan x \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i \Rightarrow \sin x$ and $\cos x$ being the two basic solutions of the homogeneous version. The old formulas give: $u' = \dfrac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \sin x \Rightarrow u(x) = -\cos x + C_1$ and $v' = \dfrac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} =$

$-\dfrac{\sin^2 x}{\cos x} = \cos x - \dfrac{1}{\cos x} \Rightarrow v(x) = \sin x - \ln\left(\dfrac{1+\sin x}{\cos x}\right) + C_2$. The final solution is thus $y = \{-\cos x \sin x + \sin x \cos x\} - \cos x \ln\left(\dfrac{1+\sin x}{\cos x}\right) + C_1 \sin x + C_2 \cos x$ [the terms inside the curly brackets cancelling out, which happens frequently in these cases].

2. $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$. Since $\lambda_{1,2} = 2 \pm 0$ [double root], the basic solutions are e^{2x} and xe^{2x}. $u' = \dfrac{\begin{vmatrix} 0 & xe^{2x} \\ \frac{e^{2x}}{x} & (1+2x)e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix}} = -1 \Rightarrow u(x) = -x + C_1$ and

$$v' = \dfrac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix}} = \frac{1}{x} \Rightarrow v(x) = \ln x + C_2.$$

Answer: $y = C_1 e^{2x} + \tilde{C}_2 x e^{2x} - xe^{2x} + \ln x \cdot xe^{2x} = e^{2x}(C_1 + C_2 x + x \ln x)$. ■

▷ **Special** cases of $r(x)$

- $r(x)$ is a **polynomial** in x:

Use a polynomial of the *same degree* but with UNDETERMINED COEFFICIENTS as a trial solution for y_p.

EXAMPLE: $y'' + 2y' - 3y = x$, $\lambda_{1,2} = 1$ and $-3 \Rightarrow y_1 = e^x$ and $y_2 = e^{-3x}$. $y_p = Ax + B$, where A and B are found by substituting this y_p into the full equation and getting: $2A - 3Ax - 3B = x \Rightarrow A = -\frac{1}{3}$ and $B = -\frac{2}{9}$.

Answer: $y = C_1 e^x + C_2 e^{-3x} - \frac{x}{3} - \frac{2}{9}$.

Exceptional case: When $\lambda = 0$, this will not work unless the trial solution y_p is further multiplied by x (when $\lambda = 0$ is a multiple root, x has to be raised to the multiplicity of λ).

EXAMPLE: $y'' - 2y' = x^2 + 1$. $\lambda_{1,2} = 0$ and $2 \Rightarrow y_1 = 1$ and $y_2 = e^{2x}$. $y_p = Ax^3 + Bx^2 + Cx$ where A, B and C are found by substituting: $6Ax + 2B - 2(3Ax^2 + 2Bx + C) = x^2 + 1 \Rightarrow A = -\frac{1}{6}$, $B = -\frac{1}{4}$ and $C = -\frac{3}{4}$.

Answer: $y = C_1 + C_2 e^{2x} - \frac{x^3}{6} - \frac{x^2}{4} - \frac{3x}{4}$. ■

- $r(x) \equiv ke^{\alpha x}$ (an **exponential** term):

The trial solution is $y_p = Ae^{\alpha x}$ [with only A to be found; this is the UNDETERMINED COEFFICIENT of this case].

EXAMPLE: $y'' + 2y' + 3y = 3e^{-2x} \Rightarrow \lambda_{1,2} = -1 \pm \sqrt{2}\, i$, $y_p = Ae^{-2x}$ substituted gives: $A(4 - 4 + 3)e^{-2x} = 3e^{-2x} \Rightarrow A = 1$.

Answer: $y_p = e^{-x}[C_1 \sin(\sqrt{2}x) + C_2 \cos(\sqrt{2}x)] + e^{-2x}$.

Exceptional case: When $\alpha = \lambda$ [any of the roots], the trial solution must be first multiplied by x (to the power of the multiplicity of this λ).

EXAMPLE: $y'' + y' - 2y = 3e^{-2x} \Rightarrow \lambda_{1,2} = 1$ and -2 [same as α]! $y_p = Axe^{-2x}$, substituted: $A(4x-4) + A(1-2x) - 2Ax = 3 \Rightarrow A = -1$ [this follows from the absolute term, the x-proportional terms cancel out, as they must].

Answer: $y = C_1 e^x + (C_2 - x)e^{-2x}$.

- $r(x) = k_s e^{px} \sin(qx)$ [or $k_c e^{px} \cos(qx)$, or a **combination** (sum) of both]:

The trial solution is $[A\sin(qx) + B\cos(qx)]e^{px}$.

EXAMPLE: $y'' + y' - 2y = 2e^{-x}\sin(4x) \Rightarrow \lambda_{1,2} = 1$ and -2 [as before]. $y_p = [A\sin(4x) + B\cos(4x)]e^{-x}$, substituted into the equation: $-18[A\sin(4x) + B\cos(4x)] - 4[A\cos(4x) - B\sin(4x)] = 2\sin(4x) \Rightarrow -18A + 4B = 2$ and $-4A - 18B = 0 \Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -18 & 4 \\ -4 & -18 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{9}{85} \\ \frac{2}{85} \end{bmatrix} \Rightarrow y = C_1 e^x + C_2 e^{-2x} + \left(\frac{2}{85}\cos(4x) - \frac{9}{85}\sin(4x)\right) e^{-2x}$.

Exceptional case: When $\lambda = p + i\,q$ [*both* the real and purely imaginary parts must agree], the trial solution acquires the standard factor of x.

'Special-case' **summary:**

We would like to mention that all these special case can be covered by *one* and the same *rule*: When $r(x) = P_n(x)e^{\beta x}$, where $P_n(x)$ is an n-degree polynomial in x, the trial solution is $Q_n(x)e^{\beta x}$, where $Q_n(x)$ is also an n-degree polynomial, but with 'undetermined' (i.e. yet to be found) coefficients.

And the same exception: When β coincides with a root of the characteristic polynomial (of multiplicity ℓ) the trial solution must be further multiplied by x^ℓ.

If we allowed complex solutions, these rules would have covered it all. Since we don't, we have to spell it out differently for $\beta = p + i\,q$:

When $r(x) = [P_s(x)\sin(qx) + P_c(x)\cos(qx)]e^{px}$ where $P_{s,c}$ are two polynomials of degree not higher than n [i.e. n is the higher of the two; also: one P may be identically equal to zero], the trial solution is: $[Q_s(x)\sin(qx) + Q_c(x)\cos(qx)]e^{px}$ with *both* $Q_{s,c}$ being polynomials of degree n [no compromise here – they both have to be there, with the full degree, even if one P is missing].

Exception: If $p + i\,q$ coincides with one of the λs, the trial solution must be further multiplied by x raised to the λ's multiplicity [note that the conjugate root $p - i\,q$ will have the same multiplicity; use the multiplicity of *one* of these – don't double it]. □

Finally, if the right hand side is a *linear combination* (sum) of such terms, we use the **superposition principle** to construct the overall y_p. This means we find y_p individually for each of the distinct terms of $r(x)$, then add them together to build the final solution.

EXAMPLE: $y'' + 2y' - 3y = x + e^{-x} \Rightarrow \lambda_{1,2} = 1, -3$. We break the right hand side into $r_1 \equiv x$ and $r_2 \equiv e^{-x}$, construct $y_{1p} = Ax + B \Rightarrow$ [when substituted into the equation with *only* x on the right hand side] $2A - 3Ax - 3B = x \Rightarrow A = -\frac{1}{3}$ and $B = -\frac{2}{9}$, and then $y_{2p} = Ce^{-x}$, substituted into the equation with r_2 only]: $C - 2C - 3C = 1 \Rightarrow C = -\frac{1}{4}$.

Answer: $y = C_1 e^x + C_2 e^{-3x} - \frac{x}{3} - \frac{2}{9} - \frac{1}{4}e^{-x}$.

Cauchy equation

looks like this:

$$(x - x_0)^2 y'' + a(x - x_0)y' + by = r(x)$$

where a, b and x_0 are specific constants (x_0 is usually equal to 0, e.g. $x^2y'' + 2xy' - 3y = x^5$).

There are two ways of solving it:

▶Converting◀

it to the previous case of a **constant-coefficient** equation (convenient when $r(x)$ is a polynomial in either x or $\ln x$ – try to figure our why). This conversion is achieved by introducing a new *independent* variable $t = \ln(x - x_0)$. We have already derived the following set of formulas for performing such a conversion: $y' = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dot{y}}{x - x_0}$ and $y'' = \frac{d^2y}{dt^2} \cdot \left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt} \cdot \frac{d^2t}{dx^2} = \frac{\ddot{y}}{(x-x_0)^2} - \frac{\dot{y}}{(x-x_0)^2}$. The original Cauchy equation thus becomes

$$\ddot{y} + (a - 1)\dot{y} + by = r(x_0 + e^t)$$

which we solve by the old technique.

EXAMPLE: $x^2y'' - 4xy' + 6y = \frac{42}{x^4} \Rightarrow \ddot{y} - 5\dot{y} + 6y = 42e^{-4t} \Rightarrow \lambda_{1,2} = 2, 3$ and $y_p = Ae^{-4t}$ substituted: $16A + 20A + 6A = 42 \Rightarrow A = 1$.

Answer: $y = C_1e^{3t} + C_2e^{2t} + e^{-4t} = C_1x^3 + C_2x^2 + \frac{1}{x^4}$ [since $x = e^t$].

Problems for extra practice

(solve by *undetermined coefficients* – *not* V of P):

1. $y'' - \dfrac{y'}{x} - \dfrac{3y}{x^2} = \ln x + 1 \Rightarrow$ [must be multiplied by x^2 first] $\ddot{y} - 2\dot{y} - 3y = te^{2t} + e^{2t} \Rightarrow$ ...

   Answer: $y = C_1x^3 + \dfrac{C_2}{x} - \left(\frac{5}{9} + \frac{\ln x}{3}\right)x^2$.

2. $x^2y'' - 2xy' + 2y = 4x + \sin(\ln x) \Rightarrow \ddot{y} - 3\dot{y} + 2y = 4e^t + \sin(t) \Rightarrow$ ...

   Answer: $y = C_1x + C_2x^2 - 4x\ln x + \frac{1}{10}\sin(\ln x) + \frac{3}{10}\cos(\ln x)$.

Warning: To use the undetermined-coefficients technique (via the t transformation) the equation must have (or must be brought to) the form of: $x^2y'' + axy' +$ ... ; to use the V-of-P technique, the equation must have the $y'' + \frac{a}{x}y' +$ ... form.

▶Directly◀

(This is more convenient when $r(x)$ is either equal to *zero*, or does *not* have the *special* form mentioned above). We substitute a *trial* solution $(x-x_0)^m$, with m yet to be determined, into the homogeneous Cauchy equation, and divide by $(x-x_0)^m$. This results in:

$$m^2+(a-1)m+b=0$$

a CHARACTERISTIC POLYNOMIAL for m. With two distinct real roots, we get our two basic solutions right away; with a duplicate root, we need an extra factor of $\ln(x-x_0)$ to construct the second basic solution; with two complex roots, we must go back to the 'conversion' technique.

EXAMPLES:

1. $x^2y''+xy'-y=0 \Rightarrow m^2-1=0 \Rightarrow m_{1,2}=\pm 1$

   Answer: $y=C_1x+C_2\dfrac{1}{x}$.

2. $x^2y''+3xy'+y=0 \Rightarrow m^2+2m+1=0 \Rightarrow m_{1,2}=-1$ (duplicate) $\Rightarrow y=\frac{C_1}{x}+\frac{C_2}{x}\ln x$.

3. $3(2x-5)^2y''-(2x-5)y'+2y=0$. First we have to rewrite it in the standard form of: $(x-\frac{5}{2})^2y''-\frac{1}{6}(x-\frac{5}{2})y'+\frac{1}{6}y=0 \Rightarrow m^2-\frac{7}{6}m+\frac{1}{6}=0 \Rightarrow m_{1,2}=\frac{1}{6},\ 1 \Rightarrow y=\tilde{C}_1(x-\frac{5}{2})+\tilde{C}_2(x-\frac{5}{2})^{1/6}=C_1(2x-5)+C_2(2x-5)^{1/6}$.

4. $x^2y''-4xy'+4y=0$ with $y(1)=4$ and $y'(1)=13$. First $m^2-5m+4=0 \Rightarrow m_{1,2}=1,\ 4 \Rightarrow y=C_1x+C_2x^4$. Then $C_1+C_2=4$ and $C_1+4C_2=13$ give $C_2=3$ and $C_1=1$.

   Answer: $y=x+3x^4$.

5. $x^2y''-4xy'+6y=x^4\sin x$. To solve the homogeneous version: $m^2-5m+6=0 \Rightarrow m_{1,2}=2,\ 3 \Rightarrow y_1=x^2$ and $y_2=x^3$. To use V-of-P formulas the equation *must be first rewritten* in the 'standard' form of $y''-\dfrac{4}{x}y'+\dfrac{6}{x^2}y=x^2\sin x \Rightarrow u'=\dfrac{\begin{vmatrix}0 & x^3\\ x^2\sin x & 3x^2\end{vmatrix}}{\begin{vmatrix}x^2 & x^3\\ 2x & 3x^2\end{vmatrix}}=-x\sin x \Rightarrow u(x)=x\cos x-\sin x+C_1$ and

   $v'=\dfrac{\begin{vmatrix}x^2 & 0\\ 2x & x^2\sin x\end{vmatrix}}{\begin{vmatrix}x^2 & x^3\\ 2x & 3x^2\end{vmatrix}}=\sin x \Rightarrow v(x)=-\cos(x)+C_2$.

   Solution: $y=(C_1-\sin x)x^2+C_2x^3$ [the rest cancelled out – common occurrence when using this technique].

Chapter 4 THIRD AND HIGHER-ORDER LINEAR ODES

First we extend the **general** linear-equation **results** to higher orders. Explicitly, we mention the third order only, but the extension to higher orders is quite obvious.

A third order **linear equation**

$$y''' + f(x)y'' + g(x)y' + h(x)y = r(x)$$

has the following general solution: $y = C_1y_1 + C_2y_2 + C_3y_3 + y_p$, where y_1, y_2 and y_3 are *three* BASIC (linearly independent) solutions of the homogeneous version. There is no general analytical technique for finding them. Should these be known (obtained by whatever other means), we can construct a PARTICULAR solution (to the full equation) y_p by the *variation of parameters* (this time we skip the details), getting:

$$u' = \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ r & y_2'' & y_3'' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}$$

with a similar formula for v' and for w' (we need three of them, one for each basic solution). The pattern of these formulas should be obvious: there is the *Wronskian* in the denominator, and the same matrix with one of its columns (the first for u', the second for v', and the last for w') replaced by $\begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$ in the numerator.

The corresponding **constant-coefficient** equation can be solved easily by constructing its *characteristic polynomial* and finding its *roots*, in a manner which is a trivial extension of the second-degree case. The main difficulty here is finding roots of higher-degree polynomials. We will take up this issue first.

Polynomial roots

▷ We start with a general **cubic polynomial** (we will call its variable x rather than λ)

$$x^3 + a_2x^2 + a_1x + a_0 = 0$$

such as, for example $x^3 - 2x^2 - x + 2 = 0$. Finding its three roots takes the following steps:

1. Compute $Q = \dfrac{3a_1 - a_2^2}{9}$ $\left[= -\frac{7}{9}\right]$ and $R = \dfrac{9a_1a_2 - 27a_0 - 2a_2^3}{54}$ $\left[= -\frac{10}{27}\right]$.

2. Now consider two cases:

   (a) $Q^3 + R^2 \geq 0$.

Compute $s = \sqrt[3]{R + \sqrt{Q^3 + R^2}}$ and $t = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$ [each being a *real* but *possibly negative* number]. The original equation has one real root given by $s + t - \frac{a_2}{3}$, and two complex roots given by $-\frac{s+t}{2} - \frac{a_2}{3} \pm \frac{\sqrt{3}}{2}(s-t)\, i$.

(b) $Q^3 + R^2 < 0$ [our case $= -\frac{1}{3}$].

Compute $\theta = \arccos \frac{R}{\sqrt{-Q^3}}$ [Q must be negative]. The equation has three real roots given by $2\sqrt{-Q}\cos\left(\frac{\theta + 2\pi k}{3}\right) - \frac{a_2}{3}$, where $k = 0$, 1 and 2. [In our case $\theta = \arccos\left(-\frac{10}{27}\sqrt{\frac{729}{343}}\right) \Rightarrow 2\sqrt{\frac{7}{9}}\cos(\frac{\theta}{3}) + \frac{2}{3} = 2$, $2\sqrt{\frac{7}{9}}\cos(\frac{\theta+2\pi}{3}) + \frac{2}{3} = -1$ and $2\sqrt{\frac{7}{9}}\cos(\frac{\theta+4\pi}{3}) + \frac{2}{3} = 1$ are the three roots].

Proof (for the s-t case only): Let x_1, x_2 and x_3 be the three roots of the formula. Expand $(x - x_1)(x - x_2)(x - x_3) = [x + \frac{a_2}{3} - (s+t)][(x + \frac{s+t}{2} + \frac{a_2}{3})^2 + \frac{3}{4}(s - t)^2] = x^3 + a_2x^2 + \left(\frac{a_2^3}{3} - 3st\right)x + \frac{a_2^3}{27} - sta_2 - s^3 - t^3$. Since $-st = Q$ and $-s^3 - t^3 = -2R$, we can see quite easily that $\frac{a_2^3}{3} - 3st = a_1$ and $\frac{a_2^3}{27} - sta_2 - s^3 - t^3 = a_0$. □

▷ And now we tackle the **forth-degree** (quartic) **polynomial**

$$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

such as, for example $x^4 + 4x^3 + 6x^2 + 4x + 1 = 0$.

1. First solve the following cubic

$$y^3 - a_2y^2 + (a_1a_3 - 4a_0)y + (4a_0a_2 - a_1^2 - a_0a_3^2) = 0$$

[$y^3 - 6y^2 + 12y - 8 = 0$ in the case of our example – the three roots are all qual to 2].

2. Then solve the following two quadratic equations

$$z^2 + \frac{1}{2}\left(a_3 \pm \sqrt{a_3^2 - 4a_2 + 4y_1}\right)z + \frac{1}{2}\left(y_1 \mp \sqrt{y_1^2 - 4a_0}\right) = 0$$

(when $2a_1 - a_3y_1 \geq 0$), or

$$z^2 + \frac{1}{2}\left(a_3 \pm \sqrt{a_3^2 - 4a_2 + 4y_1}\right)z + \frac{1}{2}\left(y_1 \pm \sqrt{y_1^2 - 4a_0}\right) = 0$$

(when $2a_1 - a_3y_1 \leq 0$), where y_1 is the largest real root of the previous cubic [$z^2 + 2z + 1 = 0$, twice, in our example]. The resulting four roots are those of the original quartic [-1, -1, -1 and -1].

Proof: We assume that $2a_1 - a_3y_1 > 0$ (the other case would be a carbon copy). By multiplying the left hand sides of the corresponding two quadratic equations one gets:

$$z^4 + a_3z^3 + a_2z^2 + \frac{a_3y_1 + \sqrt{a_3^2 - 4a_2 + 4y_1}\sqrt{y_1^2 - 4a_0}}{2}\, z + a_0$$

It remains to be shown that the linear coefficient is equal to a_1. This amounts to:

$$2a_1 - a_3y_1 = \sqrt{a_3^2 - 4a_2 + 4y_1}\sqrt{y_1^2 - 4a_0}$$

Since each of the two expressions under square root must be non-negative (see the Extra Proof below), the two sides of the equation have the same sign. It is thus legitimate to square them, obtaining a cubic equation for y_1, which is identical to the one we solved in (1). □

Extra Proof: Since y_1 is the (largest real) solution to $(a_3^2 - 4a_2 + 4y_1) \cdot (y_1^2 - 4a_0) = (2a_1 - a_3y_1)^2$, it is clear that both factors on the LHS must have the same sign. We thus have to prove that either of them is positive. Visualizing the graph of the cubic polynomial $(a_3^2 - 4a_2 + 4y_1) \cdot (y_1^2 - 4a_0) - (2a_1 - a_3y_1)^2$, it is obvious that by adding $(2a_1 - a_3y_1)^2$ to it, the largest real root can only *decrease*. This means that y_1 must be bigger than each of the real roots of $(a_3^2 - 4a_2 + 4y_1) \cdot (y_1^2 - 4a_0) = 0$, implying that $a_2 - \frac{a_3^2}{4} < y_1$ (which further implies that $4a_0 < y_1^2$). □

▷ There is **no** general **formula** for solving *fifth*-degree polynomials and beyond (investigating this led Galois to discover GROUPS); we can still find the roots *numerically* to any desired precision (the algorithms are fairly tricky though, and we will not go into this).

► There are **special cases**

of *higher-degree* polynomials which we know how to solve (or at least how to reduce them to a lower-degree polynomial). Let us mention a handful of these:

1. $x^n = a$, by finding all (n distinct) *complex* values of $\sqrt[n]{a}$, i.e. $\sqrt[n]{a}\left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right)$ when $a > 0$ and $\sqrt[n]{-a}\left(\cos\frac{2\pi k+\pi}{n} + i\sin\frac{2\pi k+\pi}{n}\right)$ when $a < 0$, both with $k = 0, 1, 2, ..., n-1$.

   Examples: (i) $\sqrt[4]{16} = 2, -2, 2i$ and $-2i$ (ii) $\sqrt[3]{-8} = -2$ and $1 \pm \sqrt{3}i$.

2. When 0 is one of the roots, it's trivial to find it, with its multiplicity.

   Example: $x^4 + 2x^3 - 4x^2 = 0$ has obviously 0 as a double root. Dividing the equation by x^2 makes it into a quadratic equation which can be easily solved.

3. When **coefficients** of an equation **add up** to 0, 1 must be one of the roots. The left hand side is divisible by $(x - 1)$ [synthetic division], which reduces its order.

   Example: $x^3 - 2x^2 + 3x - 2 = 0$ thus leads to $(x^3 - 2x^2 + 3x - 2) \div (x - 1) = x^2 - x + 2$ [quadratic polynomial].

4. When **coefficients** of the **odd** powers of x and coefficients of the **even** powers of x add up to the same two answers, then -1 is one of the roots and $(x+1)$ can be factored out.

   Example: $x^3+2x^2+3x+2=0$ leads to $(x^3+2x^2+3x+2)\div(x+1)=x^2+x+2$ and a quadratic equation.

5. Any lucky guess of a root always leads to the corresponding reduction in order (one would usually try 2, -2, 3, -3, etc. as a potential root).

6. One can cut the degree of an equation in half when the equation has **even powers** of x only by introducing $z=x^2$.

   Example: $x^4-3x^2-4=0$ thus reduces to $z^2-3z-4=0$ which has two roots $z_{1,2}=-1,\ 4$. The roots of the original equation thus are: $x_{1,2,3,4}=i,\ -i,\ 2,\ -2$.

7. Similarly with **odd powers** only.

   Example: $x^5-3x^3-4x=0$. Factoring out x [there is an extra root of 0], this becomes the previous case.

8. All powers divisible by 3 (4, 5, etc.). Use the same trick.

   Example: $x^6-3x^3-4=0$. Introduce $z=x^3$, solve for $z_{1,2}=-1,\ 4$ [same as before]. Thus $x_{1,2,3}=\sqrt[3]{-1}=-1,\ \frac{1}{2}\pm\frac{\sqrt{3}}{2}\,i$ and $x_{4,5,6}=\sqrt[3]{4},\ \sqrt[3]{4}\left(-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\,i\right)=1.5874,\ -.7937+1.3747\,i,\ =-.7937-1.3747\,i.$

9. When a **multiple root** is suspected (the question may indicate: 'there is a triple root'), the following will help: each differentiation of a polynomial reduces the multiplicity of its every root by one. This means, for example, that a triple root becomes a single root of the polynomial's *second* derivative.

   Example: $x^4-5x^3+6x^2+4x-8=0$, given there is a triple root. Differentiating twice: $12x^2-30x+12=0\Rightarrow x_{1,2}=\frac{1}{2},\ 2$. These must be substituted back into the original equation [only *one* of these is its triple root, the other is *not a root* at all]. Since 2 meets the original equation ($\frac{1}{2}$ does not), it must be its *triple* root, which can be factored out, thus: $(x^4-5x^3+6x^2+4x-8)\div(x-2)^3=x+1$ [ending up with a *linear* polynomial]. The last root is thus, trivially, equal to -1.

10. Optional: One an take a slightly more sophisticated approach when it comes to multiple roots. As was already mentioned: each differentiation of the polynomial reduces the multiplicity of every root by one, but may (and usually does) introduce a lot of extra 'phoney' roots. These can be eliminated by taking the greatest common divisor (GCD) of the polynomial and its derivative, by using EUCLID'S ALGORITHM, which works as follows:

    To find the GCD of two polynomials p and q, we divide one into the other to find the *remainder* (residue) of this operation (we are allowed to multiply the result by a constant to make it a MONIC polynomial): $r_1=\mathrm{Res}(p\div q)$, then $r_2=\mathrm{Res}(q\div r_1)$, $r_3=\mathrm{Res}(r_1\div r_2)$, ... until the remainder becomes zero. The GCD is the previous (last nonzero) r.

Example: $p(x) = x^8 - 28x^7 + 337x^6 - 2274x^5 + 9396x^4 - 24312x^3 + 38432x^2 - 33920x + 12800$. $s(x) = \text{GCD}(p, p') = x^5 - 17x^4 + 112x^3 - 356x^2 + 544x - 320$ [the r-sequence was: $x^6 - \frac{85}{4}x^4 + \frac{737}{4}x^3 - 832x^3 + 2057x^2 - 2632x + 1360$, s, 0]. $u(x) = \text{GCD}(s, s') = x^2 - 6x + 8$ [the r-sequence: $x^3 - \frac{31}{3}x^2 + 34x - \frac{104}{3}$, u, 0]. The last (quadratic) polynomial can be solved (and thus factorized) easily: $u = (x-2)(x-4)$. Thus 2 and 4 (each) must be a double root of s and a triple root of p. Taking $s \div (x-2)^2(x-4)^2 = x - 5$ reveals that 5 is a single root of s and therefore a double root of p. Thus, we have found all *eight* roots of p: 2, 2, 2, 4, 4, 4, 5 and 5. ■

Constant-coefficient equations

Similarly to solving second-order equations of this kind, we

- find the roots of the characteristic polynomial,
- based on these, construct the basic solutions of the homogeneous equation,
- find y_p by either V-P or (more commonly) undetermined-coefficient technique (which requires only a trivial and obvious extension). ■

Since the Cauchy equation is effectively a linear equation in disguise, we know how to solve it (beyond the second order) as well.

EXAMPLES:

1. $y^{iv} + 4y'' + 4y = 0 \Rightarrow \lambda^4 + 4\lambda^2 + 4 = 0 \Rightarrow z(= \lambda^2)_{1,2} = -2$ [double] $\Rightarrow \lambda_{1,2,3,4} = \pm\sqrt{2}\,i$ [both are double roots]$\Rightarrow y = C_1 \sin(\sqrt{2}x) + C_2 \cos(\sqrt{2}x) + C_3 x \sin(\sqrt{2}x) + C_4 x \cos(\sqrt{2}x)$.

2. $y''' + y'' + y' + y = 0 \Rightarrow \lambda_1 = -1$, and $\lambda^2 + 1 = 0 \Rightarrow \lambda_{2,3} = \pm i \Rightarrow y = C_1 e^{-x} + C_2 \sin x + C_3 \cos(x)$.

3. $y^v - 3y^{iv} + 3y''' - y'' = 0 \Rightarrow \lambda_{1,2} = 0$, $\lambda_3 = 1$ and $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_{4,5} = 1 \Rightarrow y = C_1 + C_2 x + C_3 e^x + C_4 x e^x + C_5 x^2 e^x$.

4. $y^{iv} - 5y'' + 4y = 0 \Rightarrow z(= \lambda^2)_{1,2} = 1,\ 4 \Rightarrow \lambda_{1,2,3,4} = \pm 2,\ \pm 1 \Rightarrow y = C_1 e^x + C_2 e^{-x} + C_3 e^{2x} + C_4 e^{-2x}$.

5. $y''' - y' = 10\cos(2x) \Rightarrow \lambda_{1,2,3} = 0,\ \pm 1$. $y_p = A\sin(2x) + B\cos(2x) \Rightarrow -10\cos(2x) + 10\sin(2x) = 10\cos(2x) \Rightarrow A = -1$, $B = 0 \Rightarrow y = C_1 + C_2 e^x + C_3 e^{-x} - \sin(2x)$.

6. $y''' - 2y'' = x^2 - 1 \Rightarrow \lambda_{1,2} = 0,\ \lambda_3 = 2$. $y_p = Ax^4 + Bx^3 + Cx^2 \Rightarrow -24Ax^2 + 24Ax - 12Bx + 6B - 4C = x^2 - 1 \Rightarrow A = -\frac{1}{24},\ B = -\frac{1}{12},\ C = \frac{1}{8} \Rightarrow y = C_1 + C_2 x + C_3 e^{2x} - \frac{x^4}{24} - \frac{x^3}{12} + \frac{x^2}{8}$.

7. $x^3 y''' + x^2 y'' - 2xy' + 2y = 0$ (homogeneous Cauchy). $m(m-1)(m-2) + m(m-1) - 2m + 2 = 0$ is the characteristic polynomial. One can readily notice $(m-1)$ being a common factor, which implies $m_1 = 1$ and $m^2 - m - 2 = 0 \Rightarrow m_{2,3} = -1,\ 2 \Rightarrow y = C_1 x + \dfrac{C_2}{x} + C_3 x^2$.

Chapter 5 SETS OF LINEAR, FIRST-ORDER, CONSTANT-COEFFICIENT ODES

First we need to complete our review of

Matrix Algebra

▶Matrix inverse & determinant◀

There are two ways of finding these

▷ The **'classroom' algorithm**

which requires $\propto n!$ number of operations and becomes not just impractical, but virtually impossible to use (even for supercomputers) when n is large (>20). Yet, for small matrices ($n \leq 4$) it's fine, and we actually prefer it. It works like this:

- In a 2×2 case, it's trivial: $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}^{-1} = \dfrac{\begin{array}{|c|c|}\hline d & -b \\ \hline -c & a \\ \hline \end{array}}{ad - bc}$ where the denominator is the determinant.

  Example: $\begin{array}{|c|c|}\hline 2 & 4 \\ \hline -3 & 5 \\ \hline \end{array}^{-1} = \begin{array}{|c|c|}\hline \frac{5}{22} & \frac{-4}{22} \\ \hline \frac{3}{22} & \frac{2}{22} \\ \hline \end{array}$, 22 being the determinant.

- The 3×3 case is done in **four steps**.

  Example: $\begin{array}{|c|c|c|}\hline 2 & 4 & -1 \\ \hline 0 & 3 & 2 \\ \hline 2 & 1 & 4 \\ \hline \end{array}^{-1}$

1. Construct a 3×3 matrix of all 2 × 2 *subdeterminants* (striking out one row and one column – organize the answers accordingly):

$$\begin{array}{|c|c|c|}\hline 10 & -4 & -6 \\ \hline 17 & 10 & -6 \\ \hline 11 & 4 & 6 \\ \hline \end{array}$$

2. *Transpose* the answer: $\begin{array}{|c|c|c|}\hline 10 & 17 & 11 \\ \hline -4 & 10 & 4 \\ \hline -6 & -6 & 6 \\ \hline \end{array}$

3. *Change* the *sign* of every other element (using the following checkerboard scheme: $\begin{array}{|c|c|c|}\hline + & - & + \\ \hline - & + & - \\ \hline + & - & + \\ \hline \end{array}$, thus: $\begin{array}{|c|c|c|}\hline 10 & -17 & 11 \\ \hline 4 & 10 & -4 \\ \hline -6 & 6 & 6 \\ \hline \end{array}$

4. *Divide by* the *determinant*, which can be obtained easily be multiplying ('scalar' product) the first row of the original matrix by the first column of the last matrix (or vice versa – one can also use the second or third row/column – column/row): $\begin{array}{|c|c|c|}\hline \frac{10}{42} & \frac{-17}{42} & \frac{11}{42} \\ \hline \frac{4}{42} & \frac{10}{42} & \frac{-4}{42} \\ \hline \frac{-6}{42} & \frac{6}{42} & \frac{6}{42} \\ \hline \end{array}$ □

If we need the **determinant only**, there is an easier scheme: $\begin{vmatrix} 2 & 4 & -1 \\ 0 & 3 & 2 \\ 2 & 1 & 4 \\ 2 & 4 & -1 \\ 0 & 3 & 2 \end{vmatrix}$ resulting in $2\times3\times4 + 0\times1\times(-1) + 2\times4\times2 - (-1)\times3\times2 - 2\times1\times2 - 4\times4\times0 = 42$.

- Essentially the same algorithm can be used for 4×4 matrices and beyond, but it becomes increasingly impractical and soon enough virtually impossible to carry out.

▷ The **'practical' algorithm**

requires $\propto n^3$ operations and can be easily converted into a computer code:

1. The original $(n\times n)$ matrix is *extended* to an $n\times 2n$ matrix by appending it with the $n\times n$ unit matrix.

2. By using one of the following **three** 'ELEMENTARY' **operations** we make the original matrix into the *unit matrix*, while the appended part results in the desired inverse:

   (a) A (full) row can be divided by any nonzero number [this is used to make the main-diagonal elements equal to 1, one by one].

   (b) A multiple of a row can be added to (or subtracted from) any other row [this is used to make the non-diagonal elements of each column equal to 0].

   (c) Two rows can be interchanged whenever necessary [when a main-diagonal element is zero, interchange the row with any *subsequent* row which has a nonzero element in that position - if none exists the matrix is SINGULAR]. □

The product of the numbers we found on the main diagonal (and had to divide by), further multiplied by -1 if there has been an *odd* number of interchanges, is the matrix' **determinant**.

- A 4×4 EXAMPLE:

$$
\begin{array}{|c|c|c|c||c|c|c|c||c}
\hline
3 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & \div 3 \\ \hline
1 & -1 & 2 & 1 & 0 & 1 & 0 & 0 & -\frac{1}{3}r_1 \\ \hline
3 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & -r_1 \\ \hline
-2 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & +\frac{2}{3}r_1 \\ \hline
\end{array}
\Rightarrow
\begin{array}{|c|c|c|c||c|c|c|c||c}
\hline
1 & 0 & \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & \\ \hline
0 & -1 & \frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & 0 & \div(-1) \\ \hline
0 & 1 & -2 & -3 & -1 & 0 & 1 & 0 & +r_2 \\ \hline
0 & 0 & -\frac{1}{3} & \frac{11}{3} & \frac{2}{3} & 0 & 0 & 1 & \\ \hline
\end{array}
$$

$$
\Rightarrow
\begin{array}{|c|c|c|c||c|c|c|c||c}
\hline
1 & 0 & \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 & 0 & 0 & +r_3 \\ \hline
0 & 1 & -\frac{5}{3} & \frac{1}{3} & \frac{1}{3} & -1 & 0 & 0 & -5r_3 \\ \hline
0 & 0 & -\frac{1}{3} & -\frac{10}{3} & -\frac{4}{3} & 1 & 1 & 0 & \div(-\frac{1}{3}) \\ \hline
0 & 0 & -\frac{1}{3} & \frac{11}{3} & \frac{2}{3} & 0 & 0 & 1 & -r_3 \\ \hline
\end{array}
\Rightarrow
\begin{array}{|c|c|c|c||c|c|c|c||c}
\hline
1 & 0 & 0 & -2 & -1 & 1 & 1 & 0 & +\frac{2}{7}r_4 \\ \hline
0 & 1 & 0 & 17 & 7 & -6 & -5 & 0 & -\frac{17}{7}r_4 \\ \hline
0 & 0 & 1 & 10 & 4 & -3 & -3 & 0 & -\frac{10}{7}r_4 \\ \hline
0 & 0 & 0 & 7 & 2 & -1 & -1 & 1 & \div 7 \\ \hline
\end{array}
$$

$$
\Rightarrow
\begin{array}{|c|c|c|c||c|c|c|c|}
\hline
1 & 0 & 0 & 0 & -\frac{3}{7} & \frac{5}{7} & \frac{5}{7} & \frac{2}{7} \\ \hline
0 & 1 & 0 & 0 & \frac{15}{7} & -\frac{25}{7} & -\frac{18}{7} & -\frac{17}{7} \\ \hline
0 & 0 & 1 & 0 & \frac{8}{7} & -\frac{11}{7} & -\frac{11}{7} & -\frac{10}{7} \\ \hline
0 & 0 & 0 & 1 & \frac{2}{7} & -\frac{1}{7} & -\frac{1}{7} & \frac{1}{7} \\ \hline
\end{array}
$$

. The last matrix is the inverse of the original matrix, as can be easily verified [no interchanges were needed]. The determinant is $3 \times (-1) \times (-\frac{1}{3}) \times 7 = 7$. ■

►Solving n equations for m unknowns◄

For an $n \times n$ *non-singular* problems with n 'small' we can use the matrix inverse: $\mathbb{A}\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = \mathbb{A}^{-1}\mathbf{b}$, but this is not very practical beyond 2×2.

▷ The fully **general technique**

which is applicable to singular as well as n by m problems works like this:

1. *Extend* $\mathbb{A}$ by an extra column $\mathbf{b}$.

2. Using 'elementary operations' attempt to make the original $\mathbb{A}$-part of the matrix into the *unit matrix* (no need to keep track of interchanges). If you succeed, the $\mathbf{b}$-part of the matrix is the (unique) solution. This of course cannot work when the number of equations and the number of unknowns don't match. Furthermore, we may run into difficulty for the following two reasons:

   (a) We may come to a column which has 0 on the main diagonal *and all elements below it* (in the same column). This column will be then skipped (as if it never existed, i.e. we will try to get 1 in the *same position* of the next column).

   (b) Discarding the columns we skipped, we may end up with fewer columns than rows [resulting in some extra rows with only zeros in their $\mathbb{A}$-part], or the other way round [resulting in some (nonzero) extra columns, which we treat in the same manner as those columns which were skipped]. The final number of 1's [on the main diagonal] is the RANK of $\mathbb{A}$.

   We will call the result of this part the MATRIX ECHELON form of the equations.

3. To *interpret* the answer we do this:

(a) If there are any 'extra' (zero $\mathbb{A}$-part) rows, we check the *corresponding* $\mathbf{b}$ elements. If they are all equal to zero, we delete the extra (redundant) rows and go to the next step; if we find even a single non-zero element among them, the original system of equations is inconsistent, and there is *no solution*.

(b) Each of the 'skipped' columns represents an unknown whose value can be chosen arbitrarily. Each row then provides an expression for one of the remaining unknowns (in terms of the 'freely chosen' ones). Note that when there are no 'skipped' columns, the solution is just a POINT in m (number of unknowns) dimensions, one 'skipped' column results in a STRAIGHT LINE, two 'skipped' columns in a PLANE, etc.

Since the first two steps of this procedure are quite straightforward, we give EXAMPLES of the interpretation part only:

1.

1	3	0	2	0	2
0	0	1	3	0	1
0	0	0	0	1	4
0	0	0	0	0	0

means that x_2 and x_4 are the 'free' parameters (often, they would be renamed c_1 and c_2, or A and B). The solution can thus be written as $\begin{cases} x_1 = 2 - 3x_2 - 2x_4 \\ x_3 = 1 - 3x_4 \\ x_5 = 4 \end{cases}$

or, in a vector-like manner:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} c_1 + \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} c_2$$

Note that this represents a (unique) plane in a five-dimensional space; the 'point' itself and the two directions (coefficients of c_1 and c_2) can be specified in infinitely many different (but equivalent) ways.

2.

1	3	0	0	0	-2	5
0	0	1	0	0	3	2
0	0	0	1	0	1	0
0	0	0	0	1	4	3

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} c_1 + \begin{bmatrix} 2 \\ 0 \\ -3 \\ -1 \\ -4 \\ 1 \end{bmatrix} c_2 \ \blacksquare$$

▶Eigenvalues & Eigenvectors◀

of a **square** matrix.

If, for a square ($n \times n$) matrix $\mathbb{A}$, we can find a *non-zero* [column] vector $\mathbf{x}$ and a (scalar) number λ such that

$$\mathbb{A}\mathbf{x} = \lambda\mathbf{x}$$

then λ is the matrix' eigenvalue and $\mathbf{x}$ is its **right eigenvector** (similarly $\mathbf{y}^T\mathbb{A} = \lambda\mathbf{y}^T$ would define its LEFT EIGENVECTOR $\mathbf{y}^T$, this time a *row* vector). This means that we seek a non-zero solutions of

$$(\mathbb{A} - \lambda\mathbb{I})\,\mathbf{x} \equiv \mathbf{0} \qquad (*)$$

which further implies that $\mathbb{A} - \lambda\mathbb{I}$ must be *singular*: $\det(\mathbb{A} - \lambda\mathbb{I}) = 0$.

The left hand side of the last equation is an n^{th}-degree polynomial in λ which has (counting multiplicity) n [possibly complex] roots. These roots are the **eigenvalues** of $\mathbb{A}$; one can easily see that for each distinct root one can find at least one right (and at least one left) eigenvector, by solving (*) for $\mathbf{x}$ (λ being known now).

It is easy to verify that

$$\begin{aligned}\det(\lambda\mathbb{I} - \mathbb{A}) \;&=\; \lambda^n - \lambda^{n-1}\cdot \mathrm{Tr}(\mathbb{A}) + \lambda^{n-2}\cdot\{\text{sum of all } 2\times 2 \text{ major subdeterminants}\}\\ &\quad - \lambda^{n-3}\cdot\{\text{sum of all } 3\times 3 \text{ major subdeterminants}\} + \dots. \pm \det(\mathbb{A})\end{aligned}$$

where $\mathrm{Tr}(\mathbb{A})$ is the sum of all main-diagonal elements. This is called the **characteristic polynomial** of $\mathbb{A}$, and its roots are the only eigenvalues of $\mathbb{A}$.

EXAMPLES:

1. $\begin{array}{|c|c|}\hline 2 & 3\\ \hline 1 & -2\\ \hline\end{array}$ has $\lambda^2 - 0\cdot\lambda - 7$ as its characteristic polynomial, which means that the eigenvalues are $\lambda_{1,2} = \pm\sqrt{7}$.

2. $\begin{array}{|c|c|c|}\hline 3 & -1 & 2\\ \hline 0 & 4 & 2\\ \hline 2 & -1 & 3\\ \hline\end{array} \Rightarrow \lambda^3 - 10\lambda^2 + (12 + 14 + 5)\lambda - 22$ [we know how to find the determinant]. The coefficients add up to 0. This implies that $\lambda_1 = 1$ and [based on $\lambda^2 - 9\lambda + 22 = 0$] $\lambda_{2,3} = \frac{9}{2} \pm \frac{\sqrt{7}}{2}i$.

3. $\begin{array}{|c|c|c|c|}\hline 2 & 4 & -2 & 3\\ \hline 3 & 6 & 1 & 4\\ \hline -2 & 4 & 0 & 2\\ \hline 8 & 1 & -2 & 4\\ \hline\end{array} \Rightarrow \lambda^4 - 12\lambda^3 + (0-4+4-4+20-16)\lambda^2 - (-64-15-28-22)\lambda + (-106) = 0$ [note there are $\binom{4}{2} = 6$ and $\binom{4}{3} = 4$ major subdeterminants of the $2{\times}2$ and 3×3 size, respectively] $\Rightarrow \lambda_1 = -3.2545$, $\lambda_2 = 0.88056$, $\lambda_3 = 3.3576$ and $\lambda_4 = 11.0163$ [these were obtained from our general formula for fourth-degree polynomials – let's hope we don't have to use it very often]. ■

The corresponding (right) **eigenvectors** can be now found by solving (*), a *homogenous* set of equations with a *singular* matrix of coefficients [therefore, there must be at least one *nonzero* solution – which, furthermore, can be multiplied by an arbitrary constant]. The number of LINEARLY INDEPENDENT (LI) solutions cannot be bigger than the *multiplicity* of the corresponding eigenvalue; establishing their correct number is an important part of the answer.

EXAMPLES:

1. Using $\mathbb{A} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & -2 \\ \hline \end{array}$ [one of our previous examples] $(\mathbb{A} - \lambda_1 \mathbb{I})\,\mathbf{x} = \mathbf{0}$ amounts to $\begin{array}{|c|c||c|}\hline 2\text{-}\sqrt{7} & 3 & 0 \\ \hline 1 & \text{-}2\text{-}\sqrt{7} & 0 \\ \hline \end{array}$, with the second equation being a multiple of the first [check it!]. We thus have to solve only $x_1 - (2 + \sqrt{7})x_2 = 0$, which has the following general solution: $\begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \end{array} = \begin{array}{|c|}\hline 2+\sqrt{7} \\ \hline 1 \\ \hline \end{array}\, c$, where c is arbitrary [geometrically, the solution represents a straight line in the x_1-x_2 plane, passing through the origin]. Any such vector, when pre-multiplied by $\mathbb{A}$, increases in length by a factor of $\sqrt{7}$, without changing direction (check it too). Similarly, replacing λ_1 by $\lambda_2 = -\sqrt{7}$, we would be getting $\begin{array}{|c|}\hline 2\text{-}\sqrt{7} \\ \hline 1 \\ \hline \end{array}\, c$ as the corresponding eigenvector. There are many equivalent ways of expressing it, $\begin{array}{|c|}\hline \text{-}3 \\ \hline 2+\sqrt{7} \\ \hline \end{array}\, \tilde{c}$ is one of them.

2. A **double eigenvalue** may possess either one or two linearly independent eigenvectors:

   (a) The unit 2×2 matrix has $\lambda = 1$ as its duplicate eigenvalue, $\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline \end{array}$ and $\begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline \end{array}$ are two LI eigenvectors [the general solution to $\begin{array}{|c|c||c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$]. This implies that *any* vector is an eigenvector of the unit matrix..

   (b) The matrix $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 0 & 1 \\ \hline \end{array}$ has the same duplicate eigenvalue of $+1$ [in general, the main diagonal elements of an UPPER-TRIANGULAR matrix are its eigenvalues], but solving $\begin{array}{|c|c||c|}\hline 0 & 2 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$ i.e. $2x_2 = 0$ has only *one* LI solution, namely $\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline \end{array}\, c$ ■

Finding eigenvectors and eigenvalues of a matrix represents the main step in solving **sets of ODE**s; we will present our further examples in that context. So let us now return to these:

Set (system) of differential equations

of *first order*, *linear*, and with *constant coefficients* typically looks like this:

$$\begin{aligned} y_1' &= 3y_1 + 4y_2 \\ y_2' &= 3y_1 - y_2 \end{aligned}$$

[the example is of the *homogeneous* type, as each term is either y_i or y_i' proportional]. The same set can be conveniently expressed in **matrix notation** as

$$\mathbf{y}' = \mathbb{A}\mathbf{y}$$

where $\mathbb{A} = \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ [both y_1 and y_2 are function of x].

for constructing a solution to any such set of n DEs is very similar to what we have seen in the case of one (linear, constant-coefficient, homogeneous) DE, namely:

We first try to find n *linearly independent* BASIC SOLUTIONS (all having the $\begin{array}{|c|}\hline y_1 \\ \hline y_2 \\ \hline \vdots \\ \hline y_n \\ \hline \end{array}$ form), then build the GENERAL SOLUTION as a linear combination (with arbitrary coefficients) of these.

It happens that the **basic solutions** can be constructed with the help of matrix algebra. To find them, we use the following *trial* solution:

$$\mathbf{y}_T = \mathbf{q} \cdot e^{\lambda x}$$

where $\mathbf{q}$ is a constant (n-dimensional) vector. Substituting into $\mathbf{y}' = \mathbb{A}\mathbf{y}$ and cancelling the (*scalar*) $e^{\lambda x}$ gives: $\lambda\mathbf{q} = \mathbb{A}\mathbf{q}$, which means λ can be any one of the *eigenvalues* of $\mathbb{A}$ and $\mathbf{q}$ be the corresponding *eigenvector*. If we find n of these (which is the case with *simple* eigenvalues) the job is done; we have effectively constructed a general solution to our set of DEs.

EXAMPLES:

1. Solve $\mathbf{y}' = \mathbb{A}\mathbf{y}$, where $\mathbb{A} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 3 & -1 \\ \hline \end{array}$. The characteristic equation is: $\lambda^2 - 2\lambda - 15 = 0 \Rightarrow \lambda_{1,2} = 1 \pm \sqrt{16} = -3$ and 5. The corresponding eigenvectors (we will call them $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$) are the solutions to $\begin{array}{|c|c||c|}\hline 6 & 4 & 0 \\ \hline 3 & 2 & 0 \\ \hline \end{array} \Rightarrow 3q_1^{(1)} + 2q_2^{(1)} = 0 \Rightarrow \mathbf{q}^{(1)} = \begin{array}{|c|}\hline 2 \\ \hline -3 \\ \hline \end{array} c_1$, and $\begin{array}{|c|c||}\hline -2 & 4 \\ \hline 3 & -6 \\ \hline \end{array}$ [from now on we will *assume* a zero right hand side]$\Rightarrow q_1^{(2)} - 2q_2^{(2)} = 0 \Rightarrow \mathbf{q}^{(2)} = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} c_2$.

   The final, general solution is thus $\mathbf{y} = c_1 \begin{array}{|c|}\hline 2 \\ \hline -3 \\ \hline \end{array} e^{-3x} + c_2 \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} e^{5x}$. Or, if you prefer, more explicitly: $\begin{array}{l} y_1 = 2c_1e^{-3x} + 2c_2e^{5x} \\ y_2 = -3c_1e^{-3x} + c_2e^{5x} \end{array}$ where c_1 and c_2 can be chosen arbitrarily.

   Often, they are specified via initial conditions, e.g. $y_1(0) = 2$ and $y_2(0) = -3$ $\Rightarrow \begin{array}{l} 2c_1 + 2c_2 = 2 \\ c_1 - 3c_2 = -3 \end{array} \Rightarrow c_1 = 1$ and $c_2 = 0 \Rightarrow \begin{array}{l} y_1 = 2e^{-3x} \\ y_2 = -3e^{-3x} \end{array}$.

2. Let us now tackle a three-dimensional problem, with $\mathbb{A} = \begin{array}{|c|c|c|}\hline 1 & -1 & 1 \\ \hline 1 & 1 & -1 \\ \hline 2 & -1 & 0 \\ \hline \end{array}$. The characteristic equation is $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0 \Rightarrow \lambda_1 = -1$ and the roots of $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_{2,3} = 1$ and 2. The respective eigenvectors are:

$\left[\begin{array}{ccc|} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|} 1 & 0 & \frac{1}{5} \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{q}^{(1)} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} c_1$, $\left[\begin{array}{ccc|} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{array}\right] \Rightarrow \mathbf{q}^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} c_2$,
and $\left[\begin{array}{ccc|} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{q}^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} c_3$. One can easily verify the correctness of each eigenvector by a simple multiplication, e.g. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \times$
$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

The general solution is thus $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} e^{-x} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{x} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2x}$.

The case of ►Double (Multiple) Eigenvalue◄

For each such eigenvalue we must first find *all* possible *solutions* of the

$$\mathbf{q}e^{\lambda x}$$

type (i.e. find all LI eigenvectors), then (if we get fewer eigenvectors than the multiplicity of λ) we have to find all possible solutions having the form of

$$(\mathbf{q}x + \mathbf{s})e^{\lambda x}$$

where $\mathbf{q}$ and $\mathbf{s}$ are two constant vectors to be found by substituting this (trial) solution into the basic equation $\mathbf{y}' = \mathbb{A}\mathbf{y}.$ As a result we get $\mathbf{q} = (\mathbb{A} - \lambda\mathbb{I})\mathbf{q}\,x + (\mathbb{A} - \lambda\mathbb{I})\mathbf{s} \;\Rightarrow \mathbf{q}$ is such a linear combination of the $\mathbf{q}$-vectors found in the previous step which allows $(\mathbb{A} - \lambda\mathbb{I})\mathbf{s} = \mathbf{q}$ to be solved in terms of $\mathbf{s}$. Note that both the *components* of $\mathbf{s}$ and the *coefficients* of the linear combination of the $\mathbf{q}$ vectors are the unknowns of this problem. One thus needs to append all the $\mathbf{q}$ vectors found in the previous step to $\mathbb{A} - \lambda\mathbb{I}$ (as extra columns) and reduces the whole (thus appended) matrix to its echelon form. It is the number of 'skipped' $\mathbf{q}$-columns which tells us how many distinct solutions there are (the 'skipped' columns $\mathbb{A} - \lambda\mathbb{I}$ would be adding, to $\mathbf{s}$, a multiple of the $\mathbf{q}e^{\lambda x}$ solution already constructed). One thus cannot get more solutions than in the previous step.

And, if still not done, we have to proceed to

$$(\mathbf{q}\frac{x^2}{2!} + \mathbf{s}x + \mathbf{u})e^{\lambda x}$$

where $\mathbf{q}$ and $\mathbf{s}$ [in corresponding pairs] is a combination of solutions from the previous step such that $(\mathbb{A} - \lambda\mathbb{I})\mathbf{u} = \mathbf{s}$ can be solved in terms of $\mathbf{u}.$ Find how many LI combinations of the $\mathbf{s}$-vectors allow a nonzero $\mathbf{u}$-solution [by the 'appending' technique], solve for the corresponding $\mathbf{u}$-vectors, and if necessary move on to the next $(\mathbf{q}\frac{x^3}{3!} + \mathbf{s}\frac{x^2}{2!} + \mathbf{u}x + \mathbf{w})e^{\lambda x}$ step, until you have as many solutions as the eigenvalue's multiplicity. Note that $(\mathbb{A} - \lambda\mathbb{I})^2\mathbf{s} = \mathbf{0}$, $(\mathbb{A} - \lambda\mathbb{I})^3\mathbf{u} = \mathbf{0}$, ...

EXAMPLES:

1. $\mathbb{A} = \begin{array}{|c|c|c|}\hline 5 & 2 & 2 \\\hline 2 & 2 & -4 \\\hline 2 & -4 & 2 \\\hline\end{array}$ has $\lambda^3 - 9\lambda^2 + 108$ as its characteristic polynomial [hint: there is a double root] $\Rightarrow 3\lambda^2 - 18\lambda = 0$ has two roots, 0 [does not check] and 6 [checks]. Furthermore, $(\lambda^3 - 9\lambda^2 + 108) \div (\lambda - 6)^2 = \lambda + 3 \Rightarrow$ the three eigenvalues are -3 and 6 [duplicate]. Using $\lambda = -3$ we get: $\begin{array}{|c|c|c||}\hline 8 & 2 & 2 \\\hline 2 & 5 & -4 \\\hline 2 & -4 & 5 \\\hline\end{array} \Rightarrow$ $\begin{array}{|c|c|c||}\hline 1 & 0 & \frac{1}{2} \\\hline 0 & 1 & -1 \\\hline 0 & 0 & 0 \\\hline\end{array} \Rightarrow c_1 \begin{array}{|c|}\hline -1 \\\hline 2 \\\hline 2 \\\hline\end{array}$, which, when multiplied by e^{-3x}, gives the first basic solution. Using $\lambda = 6$ yields: $\begin{array}{|c|c|c||}\hline -1 & 2 & 2 \\\hline 2 & -4 & -4 \\\hline 2 & -4 & -4 \\\hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 1 & -2 & -2 \\\hline 0 & 0 & 0 \\\hline 0 & 0 & 0 \\\hline\end{array} \Rightarrow c_2 \begin{array}{|c|}\hline 2 \\\hline 1 \\\hline 0 \\\hline\end{array} + c_3 \begin{array}{|c|}\hline 2 \\\hline 0 \\\hline 1 \\\hline\end{array}$ which, when multiplied by e^{6x}, supplies the remaining two basic solutions.

2. $\mathbb{A} = \begin{array}{|c|c|c|}\hline 1 & -3 & 1 \\\hline 2 & -1 & -2 \\\hline 2 & -3 & 0 \\\hline\end{array}$ has $\lambda^3 - 3\lambda - 2$ as its characteristic polynomial, with roots: $\lambda_1 = -1$ [one of our rules] $\Rightarrow (\lambda^3 - 3\lambda - 2) \div (\lambda + 1) = \lambda^2 - \lambda - 2 \Rightarrow \lambda_2 = -1$ and $\lambda_3 = 2$. So again, there is one duplicate root. For $\lambda = 2$ we get: $\begin{array}{|c|c|c||}\hline -1 & -3 & 1 \\\hline 2 & -3 & -2 \\\hline 2 & -3 & -2 \\\hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 1 & 0 & -1 \\\hline 0 & 1 & 0 \\\hline 0 & 0 & 0 \\\hline\end{array} \Rightarrow c_1 \begin{array}{|c|}\hline 1 \\\hline 0 \\\hline 1 \\\hline\end{array} e^{2x}$. For $\lambda = -1$ we get: $\begin{array}{|c|c|c||}\hline 2 & -3 & 1 \\\hline 2 & 0 & -2 \\\hline 2 & -3 & 1 \\\hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 1 & 0 & -1 \\\hline 0 & 1 & -1 \\\hline 0 & 0 & 0 \\\hline\end{array} \Rightarrow c_2 \begin{array}{|c|}\hline 1 \\\hline 1 \\\hline 1 \\\hline\end{array} e^{-x}$ [a *single* solution only]. The challenge is to construct the other (last) solution. We have to solve $\begin{array}{|c|c|c||c|}\hline 2 & -3 & 1 & 1 \\\hline 2 & 0 & -2 & 1 \\\hline 2 & -3 & 1 & 1 \\\hline\end{array} \Rightarrow \begin{array}{|c|c|c||c|}\hline 1 & 0 & -1 & \frac{1}{2} \\\hline 0 & 1 & -1 & 0 \\\hline 0 & 0 & 0 & 0 \\\hline\end{array}$, getting $\mathbf{s} = \begin{array}{|c|}\hline \frac{1}{2} \\\hline 0 \\\hline 0 \\\hline\end{array} + c \begin{array}{|c|}\hline 1 \\\hline 1 \\\hline 1 \\\hline\end{array}$, where the second part just duplicates the previous basic solution and can be discarded. The third basic solution is thus: $c_3 \left(\begin{array}{|c|}\hline 1 \\\hline 1 \\\hline 1 \\\hline\end{array} x + \begin{array}{|c|}\hline \frac{1}{2} \\\hline 0 \\\hline 0 \\\hline\end{array} \right) e^{-x} \equiv c_3 \begin{array}{|c|}\hline x + \frac{1}{2} \\\hline x \\\hline x \\\hline\end{array} e^{-x}$.

3. $\mathbb{A} = \begin{array}{|c|c|c|}\hline 42 & -9 & 9 \\\hline -12 & 39 & -9 \\\hline -28 & 21 & 9 \\\hline\end{array} \Rightarrow \lambda^3 - 90\lambda^2 + 2700\lambda - 27000$ [hint: triple root] $\Rightarrow$ $6\lambda - 180 = 0$ has a single root of 30 [$\Rightarrow$ triple root of the original polynomial]. Finding eigenvectors: $\begin{array}{|c|c|c||}\hline 12 & -9 & 9 \\\hline -12 & 9 & -9 \\\hline -28 & 21 & -21 \\\hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 1 & -\frac{3}{4} & \frac{3}{4} \\\hline 0 & 0 & 0 \\\hline 0 & 0 & 0 \\\hline\end{array} \Rightarrow c_1 \begin{array}{|c|}\hline 3 \\\hline 4 \\\hline 0 \\\hline\end{array}$ and $c_2 \begin{array}{|c|}\hline -3 \\\hline 0 \\\hline 4 \\\hline\end{array}$ are the corresponding eigenvectors [only two] which, when multiplied by e^{30x} yield the first two basic solutions. To construct the third, we set up the

equations for individual components of $\mathbf{s}$, and for the a_1 and a_2 coefficients of $\mathbf{q} \equiv a_1 \begin{array}{|c|}\hline 3\\\hline 4\\\hline 0\\\hline\end{array} + a_2 \begin{array}{|c|}\hline -3\\\hline 0\\\hline 4\\\hline\end{array}$, thus: $\begin{array}{|r|r|r||r|r|}\hline 12 & -9 & 9 & 3 & -3\\\hline -12 & 9 & -9 & 4 & 0\\\hline -28 & 21 & -21 & 0 & 4\\\hline\end{array}$. We bring this to its matrix echelon form: $\begin{array}{|c|c|c||c|c|}\hline 1 & -\frac{3}{4} & \frac{3}{4} & 0 & -\frac{1}{7}\\\hline 0 & 0 & 0 & 1 & -\frac{3}{7}\\\hline 0 & 0 & 0 & 0 & 0\\\hline\end{array}$ which first implies that $a_1 - \frac{3}{7}a_2 = 0 \Leftrightarrow a_1 = 3c_3$, $a_2 = 7c_3$. This results in $\begin{array}{|c|c|c||r|}\hline 1 & -\frac{3}{4} & \frac{3}{4} & -c_3\\\hline 0 & 0 & 0 & 0\\\hline 0 & 0 & 0 & 0\\\hline\end{array}$ to be solved for s_1, s_2 and s_3 [we need a particular solution *only*] $\Rightarrow \mathbf{s} = \begin{array}{|r|}\hline -c_3\\\hline 0\\\hline 0\\\hline\end{array}$. The third basic solution is thus $c_3 \left[\left(3\begin{array}{|c|}\hline 3\\\hline 4\\\hline 0\\\hline\end{array} + 7\begin{array}{|c|}\hline -3\\\hline 0\\\hline 4\\\hline\end{array}\right)x + \begin{array}{|c|}\hline -1\\\hline 0\\\hline 0\\\hline\end{array}\right] e^{30x} = c_3 \begin{array}{|c|}\hline -12x-1\\\hline 12x\\\hline 28x\\\hline\end{array} e^{30x}$.

4. $\mathbb{A} = \begin{array}{|r|r|r|}\hline -103 & -53 & 41\\\hline 160 & 85 & -100\\\hline 156 & 131 & -147\\\hline\end{array} \Rightarrow \lambda^3 + 165\lambda^2 + 9075\lambda + 166375$ [hint: triple root] $\Rightarrow$ $6\lambda + 330 = 0 \Rightarrow \lambda = -55$ [checks] $\Rightarrow \begin{array}{|r|r|r||}\hline -48 & -53 & 41\\\hline 160 & 140 & -100\\\hline 156 & 131 & -92\\\hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 1 & 0 & \frac{1}{4}\\\hline 0 & 1 & -1\\\hline 0 & 0 & 0\\\hline\end{array} \Rightarrow$ $c_1 \begin{array}{|r|}\hline 1\\\hline -4\\\hline -4\\\hline\end{array}$ is the only eigenvector (this, multiplied by e^{-55x}, provides the first basic solution). $\begin{array}{|r|r|r||r|}\hline -48 & -53 & 41 & 1\\\hline 160 & 140 & -100 & -4\\\hline 156 & 131 & -92 & -4\\\hline\end{array} \Rightarrow \begin{array}{|c|c|c||c|}\hline 1 & 0 & \frac{1}{4} & -\frac{9}{220}\\\hline 0 & 1 & -1 & \frac{1}{55}\\\hline 0 & 0 & 0 & 0\\\hline\end{array}$ yields $\mathbf{s} = \begin{array}{|c|}\hline -\frac{9}{220}\\\hline \frac{1}{55}\\\hline 0\\\hline\end{array}$. The second basic solution is thus $c_2(\mathbf{q}x + \mathbf{s})e^{-55x} = c_2 \begin{array}{|c|}\hline x - \frac{9}{220}\\\hline -4x + \frac{1}{55}\\\hline -4x\\\hline\end{array} e^{-55x}$. Finally, $\begin{array}{|r|r|r||c|}\hline -48 & -53 & 41 & -\frac{9}{220}\\\hline 160 & 140 & -100 & \frac{1}{55}\\\hline 156 & 131 & -92 & 0\\\hline\end{array} \Rightarrow \begin{array}{|c|c|c||c|}\hline 1 & 0 & \frac{1}{4} & -\frac{131}{48400}\\\hline 0 & 1 & -1 & \frac{39}{12100}\\\hline 0 & 0 & 0 & 0\\\hline\end{array}$ results in $\mathbf{u} = \begin{array}{|c|}\hline -\frac{131}{48400}\\\hline \frac{39}{12100}\\\hline 0\\\hline\end{array}$ and the corresponding third basic solution:

$$c_3\left(\mathbf{q}\frac{x^2}{2} + \mathbf{s}x + \mathbf{u}\right)e^{-55x} = c_3 \begin{array}{|c|}\hline \frac{x^2}{2} - \frac{9}{220}x - \frac{131}{48400}\\\hline -2x^2 + \frac{x}{55} + \frac{39}{12100}\\\hline -2x^2\\\hline\end{array} e^{-55x}. \blacksquare$$

To deal with ►Complex Eigenvalues/Vectors◄

we first write the corresponding solution in a complex form, using the regular procedure. We then replace each conjugate pair of basic solutions by the real and imaginary part (of either solution).

EXAMPLE:

$\mathbf{y}' = \boxed{\begin{array}{c|c} 2 & -1 \\ \hline 3 & 4 \end{array}} \mathbf{y} \Rightarrow \lambda^2 - 6\lambda + 11 \Rightarrow \lambda_{1,2} = 3 \pm \sqrt{2}\, i \Rightarrow \boxed{\begin{array}{c|c} -1\text{-}\sqrt{2}\, i & \text{-}1 \\ \hline 3 & 1\text{-}\sqrt{2}\, i \end{array}} \Rightarrow$ $\boxed{\begin{array}{c} 1\text{-}\sqrt{2}\, i \\ \hline \text{-}3 \end{array}}$ is the eigenvector corresponding to $\lambda_1 = 3 + \sqrt{2}\, i$ [its complex conjugate corresponds to $\lambda_2 = 3 - \sqrt{2}\, i$]. This means that the two basic solutions (in their complex form) are $\boxed{\begin{array}{c} 1 - \sqrt{2}\, i \\ \hline -3 \end{array}} e^{(3+\sqrt{2}\, i)x} \equiv \boxed{\begin{array}{c} 1 - \sqrt{2}\, i \\ \hline -3 \end{array}}$ $\left[\cos(\sqrt{2}x) + i\sin(\sqrt{2}x)\right] e^{3x}$ and its complex conjugate [$i \to -i$]. Equivalently, we can use the real and imaginary part of either of these [up to a sign, the same answer] to get: $\mathbf{y} = c_1 \boxed{\begin{array}{c} \cos(\sqrt{2}x) + \sqrt{2}\sin(\sqrt{2}x) \\ \hline -3\cos(\sqrt{2}x) \end{array}} e^{3x} +$ $c_2 \boxed{\begin{array}{c} -\sqrt{2}\cos(\sqrt{2}x) + \sin(\sqrt{2}x) \\ \hline -3\sin(\sqrt{2}x) \end{array}} e^{3x}$. This is the fully general, *real* solution to the original set of DEs. ■

Now, we extend our results to the

Non-homogeneous case

of

$$\mathbf{y}' - \mathbb{A}\mathbf{y} = \mathbf{r}(x)$$

where $\mathbf{r}$ is a given *vector* function of x (effectively n functions, one for each equation). We already know how to solve the corresponding homogeneous version.

There are two techniques to find a PARTICULAR SOLUTION $\mathbf{y}^{(p)}$ to the complete equation; the general solution is then constructed in the usual

$$c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + \dots + c_n\mathbf{y}^{(n)} + \mathbf{y}^{(p)}$$

manner.

The first of these techniques (for constructing y_p) is:

▶Variation of Parameters◀

As a *trial solution*, we use

$$\mathbf{y}^{(T)} = \mathbb{Y}\cdot\mathbf{c}$$

where $\mathbb{Y}$ is an n by n matrix of functions, with the n basic solutions of the homogeneous equation comprising its individual *columns*:

$$\mathbb{Y} \equiv \boxed{\begin{array}{c|c|c|c} \mathbf{y}^{(1)} & \mathbf{y}^{(2)} & \dots & \mathbf{y}^{(n)} \end{array}}$$

and $\mathbf{c}$ being a single column of the c_i coefficients: $\mathbf{c} \equiv \boxed{\begin{array}{c} c_1 \\ \hline c_2 \\ \hline \vdots \\ \hline c_n \end{array}}$, each now considered a *function* of x [$\mathbb{Y}\cdot\mathbf{c}$ is just a matrix representation of $c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + \dots + c_n\mathbf{y}^{(n)}$, with the c_i coefficients now being 'variable'].

Substituting in the full (non-homogeneous) equation, and realizing that $\mathbb{Y}' \equiv \mathbb{A}\cdot\mathbb{Y}$ [$\mathbb{Y}'$ represents differentiating, individually, every element of $\mathbb{Y}$] we obtain: $\mathbb{Y}'\cdot\mathbf{c}+\mathbb{Y}\cdot\mathbf{c}'-\mathbb{A}\cdot\mathbb{Y}\cdot\mathbf{c}=r\Rightarrow\mathbf{c}'=\mathbb{Y}^{-1}\cdot\mathbf{r}$. Integrating the right hand side (component by component) yields $\mathbf{c}$. The particular solution is thus

$$\mathbf{y}^{(p)}=\mathbb{Y}\int\mathbb{Y}^{-1}\cdot\mathbf{r}(x)\,dx$$

EXAMPLE:

$\mathbf{y}'=\begin{bmatrix}3&2\\1&2\end{bmatrix}\mathbf{y}+\begin{bmatrix}4e^{5x}\\0\end{bmatrix}\Rightarrow\lambda^2-5\lambda+4=0\Rightarrow\lambda_{1,2}=1$ and 4 with the respective eigenvectors [easy to construct]: $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$. Thus $\mathbb{Y}=\begin{bmatrix}e^x&2e^{4x}\\-e^x&e^{4x}\end{bmatrix}\Rightarrow$ $\mathbb{Y}^{-1}=\begin{bmatrix}\frac13e^{-x}&-\frac23e^{-x}\\\frac13e^{-4x}&\frac13e^{-4x}\end{bmatrix}$. This matrix, multiplied by $\mathbf{r}(x)$, yields $\begin{bmatrix}\frac43e^{4x}\\\frac43e^{x}\end{bmatrix}$. The componentwise integration of the last vector is trivial [the usual additive constants can be omitted to avoid duplication]: $\begin{bmatrix}\frac13e^{4x}\\\frac43e^{x}\end{bmatrix}$, (pre)multiplied by $\mathbb{Y}$ finally results in: $\mathbf{y}^{(p)}=\begin{bmatrix}3e^{5x}\\e^{5x}\end{bmatrix}$. The general solution is thus $\mathbf{y}=c_1\begin{bmatrix}1\\-1\end{bmatrix}e^x+c_2\begin{bmatrix}2\\1\end{bmatrix}e^{4x}+\begin{bmatrix}3e^{5x}\\e^{5x}\end{bmatrix}$.

Let us make this into an initial-value problem: $y_1(0)=1$ and $y_2(0)=-1\Leftrightarrow$ $\mathbf{y}(0)=\mathbb{Y}(0)\mathbf{c}+\mathbf{y}^{(p)}(0)=\begin{bmatrix}1\\-1\end{bmatrix}$. Solving for $\mathbf{c}=\begin{bmatrix}\frac13&-\frac23\\\frac13&\frac13\end{bmatrix}\left(\begin{bmatrix}1\\-1\end{bmatrix}-\begin{bmatrix}3\\1\end{bmatrix}\right)=$ $\begin{bmatrix}\frac23\\-\frac43\end{bmatrix}\Rightarrow\mathbf{y}=\begin{bmatrix}\frac23\\-\frac23\end{bmatrix}e^x-\begin{bmatrix}\frac83\\\frac43\end{bmatrix}e^{4x}+\begin{bmatrix}3e^{5x}\\e^{5x}\end{bmatrix}$. ■

The second technique for building y_p works only for two

▶Special Cases◀ of $\mathbf{r}(x)$

▷ When the non-homogeneous part of the equation has the form of

$$\left(\mathbf{a}_kx^k+\mathbf{a}_{k-1}x^{k-1}+...+\mathbf{a}_1x+\mathbf{a}_0\right)e^{\beta x}$$

we use the following 'trial' solution (which is guaranteed to work) to construct $\mathbf{y}^{(p)}$:

$$\left(\mathbf{b}_mx^m+\mathbf{b}_{m-1}x^{m-1}+...+\mathbf{b}_1x+\mathbf{b}_0\right)e^{\beta x}$$

where m equals k *plus* the multiplicity of β as an eigenvalue of $\mathbb{A}$ (if β is not an eigenvalue, $m=k$, if it is a simple eigenvalue, $m=k+1$, etc.).

When β does **not** coincide with any **eigenvalue** of $\mathbb{A}$, the equations to solve to obtain $\mathbf{b}_k$, $\mathbf{b}_{k-1}$, ..., $\mathbf{b}_1$ are

$$\begin{aligned}(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_k&=-\mathbf{a}_k\\(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k-1}&=k\mathbf{b}_k-\mathbf{a}_{k-1}\\(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k-2}&=(k-1)\mathbf{b}_{k-1}-\mathbf{a}_{k-2}\\&\vdots\\(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_0&=\mathbf{b}_1-\mathbf{a}_0\end{aligned}$$

Since $(\mathbb{A}-\beta\mathbb{I})$ is a REGULAR matrix (having an inverse), solving these is quite routine (as long as we start from the top).

When β coincides with a **simple** (as opposed to multiple) eigenvalue of $\mathbb{A}$, we have to solve

$$\begin{aligned}(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k+1} &= \mathbf{0}\\ (\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k} &= (k+1)\mathbf{b}_{k+1}-\mathbf{a}_k\\ (\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k-1} &= k\mathbf{b}_{k}-\mathbf{a}_{k-1}\\ &\vdots\\ (\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{0} &= \mathbf{b}_1-\mathbf{a}_0\end{aligned}$$

Thus, $\mathbf{b}_{k+1}$ must be the corresponding eigenvector, multiplied by such a constant as to make the second equation solvable [remember that now $(\mathbb{A}-\beta\mathbb{I})$ is *singular*]. Similarly, when solving the second equation for $\mathbf{b}_k$, a c-multiple of the same eigenvector must be added to the solution, with c chosen so that the third equation is solvable, etc. Each $\mathbf{b}_i$ is thus *unique*, even though finding it is rather tricky.

We will not try extending this procedure to the case of β being a **double** (or multiple) eigenvalue of $\mathbb{A}$.

$\triangleright$ On the other hand, the **extension** to the case of $\mathbf{r}(x)=\mathbf{P}(x)e^{px}\cos(qx)+\mathbf{Q}(x)e^{px}\sin(qx)$, where $\mathbf{P}(x)$ and $\mathbf{Q}(x)$ are polynomials in x (with vector coefficients), and $p+i\,q$ is *not* an eigenvalue of $\mathbb{A}$ is quite simple: The trial solution has the same form as $\mathbf{r}(x)$, except that the two polynomials will have UNDETERMINED coefficients, and will be of the *same* degree (equal to the degree of $\mathbf{P}(x)$ or $\mathbf{Q}(x)$, whichever is *larger*). This trial solution is then substituted into the full equation, and the coefficients of each power of x are matched, separately for the $\cos(qx)$-proportional and $\sin(qx)$-proportional terms.

In addition, one can also use the **superposition principle** [i.e. dividing $\mathbf{r}(x)$ into two or more manageable parts, getting a particular solution for each part separately, and then adding them all up].

EXAMPLES:

1. $\mathbf{y}'=\begin{bmatrix}-4&-4\\1&2\end{bmatrix}\mathbf{y}+\begin{bmatrix}1\\0\end{bmatrix}e^{2x}+\begin{bmatrix}0\\-2\end{bmatrix}e^{-x}\Rightarrow\lambda^2+2\lambda-4=0\Rightarrow\lambda_{1,2}=-1\pm\sqrt{5}.$
   We already know how to construct the solution to the homogeneous part of the equation, we show only how to deal with $\mathbf{y}^{(p)}=\mathbf{y}^{(p_1)}+\mathbf{y}^{(p_2)}$ [for each of the two $\mathbf{r}(x)$ terms]:

   $\mathbf{y}^{(p_1)}=\mathbf{b}e^{2x}$, substituted back into the equation gives $\begin{bmatrix}-6&-4\\1&0\end{bmatrix}\mathbf{b}=\begin{bmatrix}-1\\0\end{bmatrix}\Rightarrow$ $\mathbf{b}=\begin{bmatrix}0\\\frac{1}{4}\end{bmatrix}$.

   Similarly $\mathbf{y}^{(p_2)}=\mathbf{b}e^{-x}$ [a different $\mathbf{b}$], substituted, gives $\begin{bmatrix}-3&-4\\1&3\end{bmatrix}\mathbf{b}=\begin{bmatrix}0\\2\end{bmatrix}\Rightarrow$ $\mathbf{b}=\begin{bmatrix}-\frac{8}{5}\\\frac{6}{5}\end{bmatrix}$.

The full particular solution is thus $\mathbf{y}^{(p)} = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} e^{2x} + \begin{bmatrix} -\frac{8}{5} \\ \frac{6}{5} \end{bmatrix} e^{-x}$.

2. $\mathbf{y}' = \begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{x} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \Rightarrow \lambda^3 - \lambda^2 - 24\lambda - 7 = 0$. If we are interested in the particular solution *only*, we need to check that neither $\beta = 1$ nor $\beta = 0$ are the roots of the characteristic polynomial [true].

Thus $\mathbf{y}^{(p_1)} = \mathbf{b}e^{x}$ where $\mathbf{b}$ solves $\begin{bmatrix} -2 & 2 & 3 \\ 5 & -2 & -2 \\ 5 & 3 & 2 \end{bmatrix} \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{b} = \begin{bmatrix} -\frac{2}{31} \\ \frac{20}{31} \\ -\frac{25}{31} \end{bmatrix}$.

Similarly $\mathbf{y}^{(p_2)} = \mathbf{b}$ where $\begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} \Rightarrow \mathbf{b} = \begin{bmatrix} -\frac{12}{7} \\ \frac{72}{7} \\ \frac{52}{7} \end{bmatrix}$.

Answer: $\mathbf{y}^{(p)} = \begin{bmatrix} -\frac{2}{31} \\ \frac{20}{31} \\ -\frac{25}{31} \end{bmatrix} e^{x} + \begin{bmatrix} -\frac{12}{7} \\ \frac{72}{7} \\ \frac{52}{7} \end{bmatrix}$.

3. $\mathbf{y}' = \begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} x-1 \\ 2 \\ -2x \end{bmatrix}$ [characteristic polynomial same as previous example]. $\mathbf{y}^{(p)} = \mathbf{b}_1 x + \mathbf{b}_0$ with $\begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{b}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow$

$\mathbf{b}_1 = \begin{bmatrix} -\frac{5}{7} \\ \frac{51}{7} \\ -\frac{38}{7} \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{b}_0 = \begin{bmatrix} -\frac{5}{7} \\ \frac{51}{7} \\ -\frac{38}{7} \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \Rightarrow$

$\mathbf{b}_0 = \begin{bmatrix} \frac{155}{7} \\ -\frac{1210}{7} \\ \frac{863}{7} \end{bmatrix}$. Thus $\mathbf{y}^{(p)} = \begin{bmatrix} \frac{5}{7}x + \frac{155}{7} \\ \frac{51}{7}x - \frac{1210}{7} \\ -\frac{38}{7}x + \frac{863}{7} \end{bmatrix}$.

4. $\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-x} \Rightarrow \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = -1, -2$. Now our $\beta = -1$ 'coincides' with a simple eigenvalue.

$\mathbf{y}^{(p)} = (\mathbf{b}_1 x + \mathbf{b}_0)\, e^{-x}$ where $\begin{bmatrix} -3 & -3 \\ 2 & 2 \end{bmatrix} \mathbf{b}_1 = \mathbf{0} \Rightarrow$

$\mathbf{b}_1 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -3 & -3 \\ 2 & 2 \end{bmatrix} \mathbf{b}_0 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Leftrightarrow \left[\begin{array}{cc|cc} -3 & -3 & 1 & -1 \\ 2 & 2 & -1 & -2 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|cc} 1 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 8 \end{array}\right]$
.

This fixes the value of c_1 at $-8 \Rightarrow \mathbf{b}_1 = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$ and $\mathbf{b}_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + c_0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Being the last $\mathbf{b}$, we can set $c_0 = 0$ (not to duplicate the homogeneous part of the solution).

Answer: $\mathbf{y}^{(p)} = \begin{bmatrix} -8x + 3 \\ 8x \end{bmatrix} e^{-x}$.

5. $\mathbf{y}' = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 4 & -1 \\ \hline -5 & -8 & -3 \\ \hline \end{array}\,\mathbf{y} + \begin{array}{|c|}\hline x \\ \hline 0 \\ \hline 4 \\ \hline \end{array} \Rightarrow \lambda^3 - 2\lambda^2 - 15\lambda = 0 \Rightarrow \beta = 0$ is a simple eigenvalue.

We construct $\mathbf{y}^{(p)} = \mathbf{b}_2 x^2 + \mathbf{b}_1 x + \mathbf{b}_0$ where $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 4 & -1 \\ \hline -5 & -8 & -3 \\ \hline \end{array}\,\mathbf{b}_2 = \mathbf{0} \Rightarrow \mathbf{b}_2 = c_2 \begin{array}{|c|}\hline -5 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 4 & -1 \\ \hline -5 & -8 & -3 \\ \hline \end{array}\,\mathbf{b}_1 = 2\mathbf{b}_2 - \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \Leftrightarrow \begin{array}{|c|c|c||c|c|}\hline 1 & 1 & 1 & -10 & -1 \\ \hline 1 & 4 & -1 & 4 & 0 \\ \hline -5 & -8 & -3 & 6 & 0 \\ \hline \end{array} \Leftrightarrow \begin{array}{|c|c|c||c|c|}\hline 1 & 0 & \frac{5}{3} & -\frac{44}{3} & -\frac{4}{3} \\ \hline 0 & 1 & -\frac{2}{3} & \frac{14}{3} & \frac{1}{3} \\ \hline 0 & 0 & 0 & 1 & \frac{2}{15} \\ \hline \end{array}$$

$$\Rightarrow c_2 = -\tfrac{2}{15} \text{ and } \begin{array}{|c|c|c||c|}\hline 1 & 0 & \frac{5}{3} & \frac{28}{45} \\ \hline 0 & 1 & -\frac{2}{3} & -\frac{13}{45} \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} \Rightarrow \mathbf{b}_1 = \begin{array}{|c|}\hline \frac{28}{45} \\ \hline -\frac{13}{45} \\ \hline 0 \\ \hline \end{array} + c_1 \begin{array}{|c|}\hline -5 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 4 & -1 \\ \hline -5 & -8 & -3 \\ \hline \end{array}\,\mathbf{b}_0 = \mathbf{b}_1 - \begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline 4 \\ \hline \end{array} \Leftrightarrow \begin{array}{|c|c|c||c|c|}\hline 1 & 1 & 1 & -5 & \frac{28}{45} \\ \hline 1 & 4 & -1 & 2 & -\frac{13}{45} \\ \hline -5 & -8 & -3 & 3 & -\frac{180}{45} \\ \hline \end{array} \Leftrightarrow \begin{array}{|c|c|c||c|c|}\hline 1 & 0 & \frac{5}{3} & -\frac{22}{3} & \frac{125}{135} \\ \hline 0 & 1 & -\frac{2}{3} & \frac{7}{3} & -\frac{41}{135} \\ \hline 0 & 0 & 0 & -15 & -\frac{81}{45} \\ \hline \end{array}$$

$$\Rightarrow c_1 = -\tfrac{3}{25} \text{ and } \begin{array}{|c|c|c||c|}\hline 1 & 0 & \frac{5}{3} & \frac{1219}{675} \\ \hline 0 & 1 & -\frac{2}{3} & -\frac{394}{675} \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} \Rightarrow \mathbf{b}_0 = \begin{array}{|c|}\hline \frac{1219}{675} \\ \hline -\frac{394}{675} \\ \hline 0 \\ \hline \end{array} \text{ [no need for } c_0\text{].}$$

Answer: $\mathbf{y}^{(p)} = \begin{array}{|c|}\hline \frac{2}{3}x^2 + \frac{11}{9}x + \frac{1219}{675} \\ \hline -\frac{4}{15}x^2 - \frac{119}{225}x - \frac{394}{675} \\ \hline -\frac{2}{5}x^2 - \frac{9}{25}x \\ \hline \end{array}$.

▶Two Final Remarks◀

Note that an equation of the type

$$\mathbb{B}\mathbf{y}' = \mathbb{A}\mathbf{y} + \mathbf{r}$$

can be converted to the regular type by (pre)multiplying it by $\mathbb{B}^{-1}$.

EXAMPLE:

$$\begin{array}{r} 2y_1' - 3y_2' = y_1 - 2y_2 + x \\ y_1' + y_2' = 2y_1 + 3y_2 - 4 \end{array} \quad \Leftrightarrow \quad \begin{array}{l} y_1' = \frac{7}{5}\,y_1 + \frac{7}{5}\,y_2 + \frac{1}{5}\,x - \frac{12}{5} \\ y_2' = \frac{3}{5}\,y_1 + \frac{8}{5}\,y_2 - \frac{1}{5}\,x - \frac{8}{5} \end{array}$$

which we know how to solve.

And, **finally**: Many important systems of differential equations you encounter in Physics are *nonlinear*, of *second* order. A good example is: $\ddot{\mathbf{r}} + \mu\frac{\mathbf{r}}{r^3} = 0$ (μ is a constant) describing motion of a planet around the sun. Note that $\mathbf{r}$ actually stands for *three dependent* variables, $x(t)$, $y(t)$ and $z(t)$. There is no general method for solving such equations, specialized techniques have to be developed for each particular case. We cannot pursue this topic here.

Chapter 6 POWER-SERIES SOLUTION

of the following equation

$$y'' + f(x)y' + g(x)y = 0 \qquad \text{(MAIN)}$$

So far we have not discovered any general procedure for solving such equations. The technique we develop in this chapter can do the trick, but it provides the MACLAURIN EXPANSION of the solution *only* (which, as we will see, is quite often sufficient to identify the function).

We already know that once the solution to a homogeneous equation is found, the V of P technique can easily deal with the corresponding nonhomogeneous case (e.g. any $r(x)$ on the right hand side of MAIN). This is why, in this chapter, we restrict our attention to *homogeneous equations only*.

The main idea

is to express y as a **power series** in x, with yet to be determined coefficients:

$$y(x) = \sum_{i=0}^{\infty} c_i x^i$$

then substitute this expression into the differential equation to be solved [$y' = \sum_{i=1}^{\infty} i c_i x^{i-1}$ needs to be multiplied by $f(x)$ and $y'' = \sum_{i=2}^{\infty} i(i-1)c_i x^{i-2}$ by $g(x)$, where both $f(x)$ and $g(x)$ must be expanded in the same manner – they are usually in that form already] and make the *overall* coefficient of each power of x is equal to zero.

This results in (infinitely many, but regular) equations for the unknown coefficients c_0, c_1, c_2, ... These can be solved in a RECURRENT [some call it RECURSIVE] manner (i.e. by deriving a simple formula which computes c_k based on c_0, c_1, ..., c_{k-1}; c_0 and c_1 can normally be chosen *arbitrarily*).

EXAMPLES:

1. $y'' + y = 0 \Rightarrow \sum_{i=2}^{\infty} i(i-1)c_i x^{i-2} + \sum_{i=0}^{\infty} c_i x^i \equiv 0$. The main thing is to express the left hand side as a *single* infinite summation, by replacing the index i of the first term by i^*+2, thus: $\sum_{i^*=0}^{\infty}(i^*+2)(i^*+1)c_{i^*+2}x^{i^*}$ [note that the lower limit had to be adjusted accordingly]. But i^* is just a dummy index which can be called j, k or anything else *including* i. This way we get (combining both terms): $\sum_{i=0}^{\infty}\left[(i+2)(i+1)c_{i+2} + c_i\right]x^i \equiv 0$ which implies that the expression in square brackets must be identically equal to zero. This yields the following **recurrent formula**

$$c_{i+2} = \frac{-c_i}{(i+2)(i+1)}$$

where $i = 0, 1, 2, ...$, from which we can easily construct the compete sequence of the c-coefficients, as follows: Starting with c_0 arbitrary, we get

$c_2 = \frac{-c_0}{2\times 1}$, $c_4 = \frac{-c_2}{4\times 3} = \frac{c_0}{4!}$, $c_6 = \frac{-c_4}{6\times 5} = \frac{-c_0}{6!}$, ..., $c_{2k} = \frac{(-1)^k}{(2k)!}$ in *general*. Similarly, choosing an arbitrary value for c_1 we get $c_3 = \frac{-c_1}{3!}$, $c_5 = \frac{c_1}{5!}$, ... The complete solution is thus

$$y = c_0(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...) + c_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...)$$

where the infinite expansions can be easily identified as those of $\cos x$ and $\sin x$, respectively. We have thus obtained the expected $y = c_0 \cos x + c_1 \sin x$ [check].

We will not always be lucky enough to identify each solution as a combination of simple functions, but do **learn to recognize** at least the following expansions:

$$\begin{aligned} (1-ax)^{-1} &= 1 + ax + a^2x^2 + a^3x^3 + ... \\ e^{ax} &= 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + ... \\ \ln(1-ax) &= -ax - \frac{a^2x^2}{2} - \frac{a^3x^3}{3} - ... \qquad \text{(no factorials)} \end{aligned}$$

with a being any number [often $a = 1$].

And **realize** that $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...$ [a power of x missing] must be $\frac{\sin x}{x}$, $1 - \frac{3x^2}{2!} + \frac{9x^4}{4!} - \frac{27x^6}{6!} + ...$ is the expansion of $\cos(\sqrt{3}x)$, $1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + ...$ must be $\exp(x^2)$, and $1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + ...$ is $\frac{\sin\sqrt{x}}{\sqrt{x}}$.

2. $(1-x^2)y'' - 2xy' + 2y = 0 \Rightarrow \sum\limits_{i=2}^{\infty} i(i-1)c_i x^{i-2} - \sum\limits_{i=2}^{\infty} i(i-1)c_i x^i - 2\sum\limits_{i=1}^{\infty} ic_i x^i + 2\sum\limits_{i=0}^{\infty} c_i x^i \equiv 0$. By reindexing (to get the same x^i in each term) we get $\sum\limits_{i^*=0}^{\infty}(i^* + 2)(i^*+1)c_{i^*+2}x^{i^*} - \sum\limits_{i=2}^{\infty} i(i-1)c_i x^i - 2\sum\limits_{i=1}^{\infty} ic_i x^i + 2\sum\limits_{i=0}^{\infty} c_i x^i \equiv 0$. Realizing that, as a dummy index, i^* can be called i (this is the last time we introduced i^*, form now on we will call it i directly), our equation becomes:

$$\sum_{i=0}^{\infty} \left[(i+2)(i+1)c_{i+2} - i(i-1)c_i - 2ic_i + 2c_i\right] x^i \equiv 0$$

[we have adjusted the lower limit of the second and third term down to 0 *without* affecting the answer – careful with this though, things are not always that simple]. The square brackets must be identically equal to zero which implies:

$$c_{i+2} = \frac{i^2 + i - 2}{(i+2)(i+1)} c_i = \frac{i-1}{i+1} c_i$$

where $i = 0, 1, 2,$ .... Starting with an arbitrary c_0 we get $c_2 = -c_0$, $c_4 = \frac{1}{3}c_2 = -\frac{1}{3}c_0$, $c_6 = \frac{3}{4}c_4 = -\frac{1}{4}c_0$, $c_6 = -\frac{1}{6}c_0$, ... Starting with c_1 we get $c_3 = 0$, $c_5 = 0$, $c_7 = 0$, ... The solution is thus

$$y = c_0(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - ...) + c_1 x$$

One of the basic solutions is thus simply equal to x, once we know that we can use the V of P technique to get an *analytic expression* for the other solution, thus: $y^T(x) = c(x) \cdot x$ substituted into the original equation gives:

$$c''x(1-x^2) + 2c'(1-2x^2) = 0$$

With $c' = z$ this gives $\frac{dz}{z} = -2\frac{(1-2x^2)}{x(1-x^2)} = -2\left[\frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}\right]$ (partial fractions). To solve for A, B and C we substitute 0, 1 and -1 into $A(1-x^2) + Bx(1+x) + Cx(1-x) = 0 \Rightarrow A = 1$, $B = -\frac{1}{2}$ and $C = \frac{1}{2}$. Thus $\ln z = -2\left[\ln x + \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)\right] + \tilde{c} \Rightarrow z = \frac{\hat{c}}{x^2(1-x^2)} = \hat{c}\left[\frac{\bar{A}}{x^2} + \frac{\bar{B}}{x} + \frac{\bar{C}}{1-x} + \frac{\bar{D}}{1+x}\right] =$ (after solving in a similar manner) $\hat{c}\left[\frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)}\right] \Rightarrow$ $c(x) = \hat{c}\left(-\frac{1}{x} + \frac{1}{2}\ln\frac{1+x}{1-x}\right) + c_1 \Rightarrow y^T = c_0\left(1 - \frac{x}{2}\ln\frac{1+x}{1-x}\right) + c_1 x$. One can easily verify that this agrees with the previous expansion.

3. $y'' - 3y' + 2y = 0 \Rightarrow \sum\limits_{i=2}^{\infty} i(i-1)c_i x^{i-2} - 3\sum\limits_{i=1}^{\infty} ic_i x^{i-1} + 2\sum\limits_{i=0}^{\infty} c_i x^i \equiv 0 \Rightarrow$ $\sum\limits_{i=0}^{\infty}\left[(i+2)(i+1)c_{i+2} - 3(i+1)c_{i+1} + 2c_i\right]x^i \equiv 0 \Rightarrow c_{i+2} = \frac{3(i+1)c_{i+1} - 2c_i}{(i+2)(i+1)}$. By choosing $c_0 = 1$ and $c_1 = 0$ we can generate the *first* basic solution [$c_2 = \frac{3\times 0 - 2\times 1}{2} = -1$, $c_3 = \frac{3\times 2\times(-1) - 2\times 0}{3\times 2} = -1$, ...]:

$$c_0(1 - x^2 - x^3 - \frac{7}{12}x^4 - \frac{1}{4}x^5 - ...)$$

similarly with $c_0 = 0$ and $c_1 = 1$ the *second* basic solution is:

$$c_1(x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{5}{8}x^4 + \frac{31}{120}x^5 + ...)$$

There is no obvious pattern to either sequence of coefficients. Yet we know that, in this case, the two basic solutions should be simply e^x and e^{2x}. The trouble is that our power-series technique presents these in a hopelessly entangled form of $2e^x - e^{2x}$ [*our* first basic solution] and $e^{2x} - e^x$ [the second], and we have no way of properly separating them.

Sometimes the *initial conditions* may help, e.g. $y(0) = 1$ and $y'(0) = 1$ [these are effectively the values of c_0 and c_1, respectively], leading to $c_2 = \frac{3-2}{2} = \frac{1}{2}$, $c_3 = \frac{3-2}{3\times 2} = \frac{1}{6}$, $c_4 = \frac{\frac{3}{2}-1}{4\times 3} = \frac{1}{24}$, ... form which the pattern of the e^x-expansion clearly emerges. We can then *conjecture* that $c_i = \frac{1}{i!}$ and *prove it* by substituting into $(i+2)(i+1)c_{i+2} - 3(i+1)c_{i+1} + 2c_i = (i+2)(i+1)\frac{1}{(i+2)!} - 3(i+1)\frac{1}{(i+1)!} + 2\frac{1}{i!} = \frac{1-3+2}{i!} \equiv 0$. Similarly, the initial values of $y(0) = c_0 = 1$ and $y'(0) = c_1 = 2$ will lead to $1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{3!} + ...$ [the expansion of e^{2x}]. Prove that $c_i = \frac{2^i}{i!}$ is also a solution of our recurrence equation!

In a case like this, I often choose such 'helpful' initial conditions; if not, you would be asked to present the first five nonzero terms of each basic (entangled) solution only (without identifying the function). ■

Optional (up to the $\otimes$ mark):

Sturm-Liouville eigenvalue problem

The basic equation of this chapter, namely $y'' + f(x)y' + g(x)y = 0$ can be always rewritten in the form of

$$[p(x)y']' + q(x)y = 0$$

where $p(x) = e^{\int f(x)\,dx}$ and $q(x) = g(x) \cdot p(x)$ [verify]. In Physics, we often need to solve such an equation under the following simple conditions of $y(x_1) = y(x_2) = 0$. A trivial solution to this boundary-value problem is always $y \equiv 0$, the real task is to find *non-zero* solutions, if possible.

Such nonzero solutions will be there, but only under rather *special* circumstances. To give ourselves more flexibility (and more chances of finding a nonzero solution), we multiply the q-function by an arbitrary number, say $-\lambda$, to get: $(py')' = \lambda qy$, or equivalently

$$\frac{(py')'}{q} = \lambda y \qquad \text{(S-L)}$$

One can easily verify that $\dfrac{(py')'}{q} \equiv \mathcal{A}y$ defines a **linear operator** [an OPERATOR is a 'prescription' for modifying a function into another function; linear means $\mathcal{A}(c_1y_1 + c_2y_2) = c_1\mathcal{A}y_1 + c_2\mathcal{A}y_2$, where c_1 and c_2 are arbitrary *constants*].

The situation is now quite similar to our old eigenvalue problem of $\mathbb{A}\mathbf{y} = \lambda\mathbf{y}$, where $\mathbb{A}$ is also a linear operator, but acting on *vectors* rather than *functions*. The analogy is quite appropriate: similarly to the matrix/vector case we can find a nonzero solution to (S-L) only for *some* values of λ, which will also be called the problem's EIGENVALUES [the corresponding y's will be called EIGENFUNCTIONS]. The only major difference is that now we normally find *infinitely many* of these.

One can also prove that, for two *distinct* eigenvalues, the corresponding eigenfunctions are ORTHOGONAL, in the following sense:

$$\int_{x_1}^{x_2} q(x) \cdot y_1(x) \cdot y_2(x)\,dx = 0$$

Proof: By our assumptions: $\begin{cases} (py_1')' = \lambda_1 y_1 q \\ (py_2')' = \lambda_2 y_2 q \end{cases}$. Multiply the first equation by y_2 and the second one by y_1 and subtract, to get: $y_2(py_1')' - y_1(py_2')' = (\lambda_1 - \lambda_2)q\,y_1\,y_2$. Integrate this (the left hand side by parts) from x_1 to x_2: $y_2py_1'|_{x_1}^{x_2} - y_1py_2'|_{x_1}^{x_2} = (\lambda_1 - \lambda_2)\int_{x_1}^{x_2} q\,y_1\,y_2\,dx$. The left hand side is zero due to our boundary conditions, which implies the rest. $\square$

Notes:

- When $p(x_1) = 0$, we can *drop* the *initial condition* $y(x_1) = 0$ [same with $p(x_2) = 0$].

- We *must* always insist that each solution *be integrable* in the $\int\limits_{x_1}^{x_2} q\,y^2\,dx$ sense [to have a true eigenvalue problem]. From now on, we allow only such integrable functions as solutions to a S-L problem, without saying.
- Essentially the same proof would hold for a slightly more complicated equation
$$(py')' + r\,y = \lambda\,q\,y$$
where r is yet another specific function of x (we are running out of letters – nothing to do with the old nonhomogeneous term, also denoted r). ■

EXAMPLES:

1. $[(1-x^2)y']' + \lambda y = 0$ is a so called LEGENDRE equation. Its (integrable, between $x_1 = -1$ and $x_2 = 1$) solutions must meet $\int\limits_{-1}^{1} y_1(x)\cdot y_2(x)\,dx = 0$ [since $1-x^2 = 0$ at each x_1 and x_2, we don't need to impose any boundary conditions on y].
2. $[\sqrt{1-x^2}y']' + \dfrac{\lambda y}{\sqrt{1-x^2}} = 0$ is the CHEBYSHEV equation. The solutions meet $\int\limits_{-1}^{1} \frac{y_1(x)\cdot y_2(x)}{\sqrt{1-x^2}}\,dx = 0$ [no boundary conditions necessary].
3. $[xe^{-x}y']' + \lambda e^{-x}y = 0$ is the LAGUERRE equation ($x_1 = 0$, $x_2 = \infty$). The solutions are orthogonal in the $\int\limits_{0}^{\infty} e^{-x}\,y_1\,y_2\,dx = 0$ sense [no boundary conditions necessary]. ⊗

Using our power-series technique, we are able to solve the above equations (and, consequently, the corresponding eigenvalue problem). Let us start with the

▶Legendre Equation◀

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

(Note that we already solved this equation with $\lambda = 2$, see Example 2 of the 'main idea' section).

The expression to be identically equal to zero is $(i+2)(i+1)c_{i+2} - i(i-1)c_i - 2ic_i + \lambda c_i \Rightarrow c_{i+2} = -\dfrac{\lambda - (i+1)\,i}{(i+2)(i+1)}c_i$. If we allow the c_i-sequence to be infinite, the corresponding function is *not* integrable in the $\int\limits_{-1}^{1} y^2\,dx$ sense [we skip showing that, they do it in Physics], that is why we have to insist on finite, i.e. *polynomial* solution. This can be arranged only if the numerator of the $c_{i+2} = \ldots$ formula is zero for some integer value of i, i.e. iff

$$\lambda = (n+1)\,n$$

[these are the **eigenvalues** of the corresponding S-L problem].

We then get the following polynomial solutions – $P_n(x)$ being the standard notation:

$$\begin{aligned} P_0(x) &\equiv 1 \\ P_1(x) &= x \\ P_2(x) &= 1-3x^2 \\ P_3(x) &= x-\frac{5}{3}x^3 \\ P_4(x) &= 1-10x^2+\frac{35}{3}x^4 \\ &\quad .... \end{aligned}$$

i.e., in general, $P_n(x) = 1 - \frac{n\cdot(n+1)}{2!}x^2 + \frac{(n-2)n\cdot(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n\cdot(n+1)(n+3)(n+5)}{6!}x^6 +$ ... when n is even, and $P_n(x) = x - \frac{(n-1)\cdot(n+2)}{3!}x^3 + \frac{(n-3)(n-1)\cdot(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)\cdot(n+2)(n+4)(n+6)}{7!}x^7 +$ ... when n is odd.

Realize that, with each new value of λ, the corresponding P_n-polynomial solves a slightly *different* equation (that is why we have so many solutions). But we also know that, for each n, the (single) equation should have a *second* solution. It does, but the form of these is somehow more complicated, namely: $Q^{(1)}_{n-1} + Q^{(2)}_n \cdot \ln\frac{1+x}{1-x}$, where $Q^{(1)}_{n-1}$ and $Q^{(2)}_n$ are polynomials of degree $n-1$ and n, respectively (we have constructed the full solution for the $n = 1$ case). These so called LEGENDRE FUNCTIONS OF SECOND KIND are of lesser importance in Physics [since they don't meet the integrability condition, they don't solve the eigenvalue problem]. We will not go into further details.

Optional: ►Associate Legendre Equation◄

$$(1-x^2)y'' - 2xy' + \left[(n+1)n - \frac{m^2}{1-x^2}\right]y = 0$$

can be seen as an eigenvalue problem with $q(x) = \frac{1}{1-x^2}$ and $\lambda = m^2$ [the solutions will thus be orthogonal in the $\int\limits_{-1}^{1}\frac{y_1\cdot y_2}{1-x^2}\,dx = 0$ sense, assuming that they share the same n but the m's are different].

By using the substitution $y(x) = (1-x^2)^{m/2}\cdot u(x)$, the equation is converted to $(1-x^2)u'' - 2(m+1)xu' + [(n+1)n - (m+1)m]u = 0$, which has the following polynomial solution: $P^{(m)}_n(x)$ [the m^{th} derivative of the Legendre polynomial P_n].

Proof: Differentiate the Legendre equation m times: $(1-x^2)P''_n - 2xP'_n + (n+1)nP_n = 0$ getting: $(1-x^2)P^{(m+2)}_n - 2mxP^{(m+1)}_n - m(m-1)P^{(m)}_n - 2xP^{(m+1)}_n - 2mP^{(m)}_n + (n+1)nP^{(m)} = (1-x^2)P^{(m+2)}_n - 2x(m+1)P^{(m+1)}_n - (m+1)mP^{(m)}_n + (n+1)nP^{(m)}_n = 0$ □ ⊗

►Chebyshev equation◄

$$(1-x^2)y'' - xy' + \lambda y = 0$$

[verify that this is the *same* equation as in the S-L discussion].

It easy to see that the Legendre recurrence formula needs to be modified to read $(i+2)(i+1)c_{i+2} - i(i-1)c_i - ic_i + \lambda c_i = 0$ [only the ic_i coefficient has changed from -2 to -1] $\Rightarrow c_{i+2} = -\dfrac{\lambda - i^2}{(i+2)(i+1)} c_i$. To make the expansion finite [and the resulting function square-integrable] we have to choose

$$\lambda = n^2 \qquad \text{(Eigenvalues)}$$

This leads to the following Chebyshev polynomials:

$$\begin{aligned} T_0 &\equiv 1 \\ T_1 &= x \\ T_2 &= 1 - 2x^2 \\ T_3 &= x - \frac{4}{3}x^3 \\ &.... \end{aligned}$$

i.e. $1 - \frac{n^2}{2!}x^2 + \frac{n^2(n^2-2^2)}{4!}x^4 - \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}x^6 + ...$ in the even case, and $x - \frac{n^2-1}{3!}x^3 + \frac{(n^2-1)(n^2-3^2)}{5!}x^5 - \frac{(n^2-1)(n^2-3^2)(n^2-5^2)}{7!}x^7 + ...$ in the odd case.

The corresponding set of *second* basic solutions would consist of functions of the $\sqrt{1-x^2}Q_n(x)$ type, where Q_n is also a polynomial of degree n.

Method of Frobenius

The power-series technique described so far is applicable only when both $f(x)$ and $g(x)$ of the MAIN equation can be expanded at $x = 0$. This condition is violated when either f or g (or both) involve a division by x or its power [e.g. $y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})\, y = 0$].

To make the power-series technique work in some of these cases, we must extend it in a manner described shortly (the extension is called the method of FROBENIUS). The new restriction is that the singularity of f is (at most) of the first degree in x, and that of g is no worse than of the second degree. We can thus rewrite the main equation as

$$y'' + \frac{a(x)}{x}y' + \frac{b(x)}{x^2}y = 0 \qquad \text{(Frobenius)}$$

where $a(x)$ and $b(x)$ are REGULAR [i.e. 'expandable': $a(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$, and $b(x) = b_0a_0 + b_1x + b_2x^2 + b_3x^3 + ...$].

The **trial solution** now has the form of

$$y^{(T)} = \sum_{i=0}^{\infty} c_i x^{r+i} = c_0x^r + c_1x^{r+1} + c_2x^{r+2} + ...$$

where r is a *number* (no necessarily an integer) yet to be found. When substituted into the above differential equation (which is normally simplified by multiply it by x^2), the overall coefficient of the lowest (r^{th}) power of x is $[r(r-1) + a_0r + b_0]\, c_0$.

This must (as all the other coefficients) be equal to zero, yielding the so called **indicial equation** for r

$$r^2 + (a_0 - 1)r + b_0 = 0$$

Even after ignoring the possibility of complex roots [assume this never happens to us], we have to categorize the solution of the indicial (simple quadratic) equation into **three** separate **cases**:

1. *Two distinct* real roots which don't differ by an integer
2. A *double* root
3. Two roots which *differ* by an *integer*, i.e. $r_2 - r_1$ is a nonzero integer (zero is covered by Case 2). ■

We have to develop our technique separately for each of the three cases:

▶Distinct Real Roots◀

The trial solution is substituted into the differential equation with r having the value of one of the roots of the indicial equation. Making the coefficients of each power of x cancel out, one gets the usual recurrence formula for the sequence of the c-coefficients [this time we get *two* such *sequences*, one with the first root r_1 and the other, say c_i^*, with r_2; this means that we don't have to worry about intermingling the two basic solutions – the technique now automatically separates them for us]. Each of the two recurrence formula allows a free choice of the first c (called c_0 and c_0^*, respectively); the rest of each sequence must uniquely follow.

EXAMPLE:

$x^2y'' + (x^2 + \frac{5}{36})y = 0$ [later on we will see that this is a special case of the so called BESSEL EQUATION]. Since $a(x) \equiv 0$ and $b(x) = x^2 + \frac{5}{36}$ the indicial equation reads $r^2 - r + \frac{5}{36} = 0 \Rightarrow r_{1,2} = \frac{1}{6}$ and $\frac{5}{6}$ [Case 1]. Substituting our trial solution into the differential equation yields $\sum\limits_{i=0}^{\infty} c_i(r+i)(r+i-1)x^{r+i} + \frac{5}{36}\sum\limits_{i=0}^{\infty} c_i x^{r+i} + \sum\limits_{i=0}^{\infty} c_i x^{r+i+2} = 0$. Introducing a new dummy index $i^* = i+2$ we get $\sum\limits_{i=0}^{\infty} c_i[(r+i)(r+i-1)+\frac{5}{36}]x^{r+i} + \sum\limits_{i^*=2}^{\infty} c_{i^*-2}x^{r+i^*} = 0$ [as always, i^* can now be replaced by i]. Before we can combine the two sums together, we have to deal with the exceptional $i = 0$ and 1 terms. The first ($i = 0$) term gave us our indicial equation and was made to disappear by taking r to be one of the equation's two roots. The second one has the coefficient of $c_1[(r+1)r + \frac{5}{36}]$ which can be eliminated only by $c_1 \equiv 0$. The rest of the left hand side is $\sum\limits_{i=0}^{\infty} \left\{c_i[(r+i)(r+i-1) + \frac{5}{36}] + c_{i-2}\right\} x^{r+i} \Rightarrow c_i = \dfrac{-c_{i-2}}{(r+i)(r+i-1)+\frac{5}{36}}$. So far we have avoided substituting a specific root for r [to be able to deal with both cases at the same time], now, to build our two basic solutions, we have to set

1. $r = \frac{1}{6}$, getting $c_i = \dfrac{-c_{i-2}}{i\,(i-\frac{2}{3})} \Rightarrow c_2 = \frac{-c_0}{2\times\frac{4}{3}}$, $c_4 = \frac{c_0}{4\times 2\times\frac{4}{3}\times\frac{10}{3}}$, $c_6 = \frac{-c_0}{6\times 4\times 2\times\frac{4}{3}\times\frac{10}{3}\times\frac{16}{3}}$, ... [the odd-indexed coefficients must be all equal to zero]. Even though the expansion has an obvious pattern, the function cannot be identified as a 'known'

function. Based on this expansion, one can *introduce* a new function [eventually a whole set of them], called BESSEL, as we do in full detail later on. The first basic solution is thus $y_1 = c_0 x^{\frac{1}{6}}(1 - \frac{3}{8}x^2 + \frac{9}{320}x^4 - \frac{9}{10240}x^6 + ...)$. [For those of you who know the Γ-function, the solution can be expressed in a more compact form of $\tilde{c} \sum\limits_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+1/6}}{k!\Gamma(k+\frac{2}{3})}$].

2. $r = \frac{5}{6}$, getting $c_i^* = \dfrac{-c_{i-2}^*}{i\,(i+\frac{2}{3})} \Rightarrow c_2^* = \frac{-c_0^*}{2\times\frac{8}{3}}$, $c_4^* = \frac{c_0^*}{4\times2\times\frac{8}{3}\times\frac{14}{3}}$, $c_6^* = \frac{-c_0^*}{6\times4\times2\times\frac{8}{3}\times\frac{14}{3}\times\frac{20}{3}}$, ... $\Rightarrow$
$y_2 = c_0^* x^{\frac{5}{6}}(1 - \frac{3}{16}x^2 + \frac{9}{896}x^4 - \frac{9}{35840}x^6 + ...)$ [$= \tilde{c}^* \sum\limits_{k=0}^{\infty} \frac{(-1)^k(\frac{x}{2})^{2k+5/6}}{k!\Gamma(k+\frac{4}{3})}$].

▶Double root◀

The first basic solution is constructed in the usual manner of $y_1 = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + ...$ The second basic solution has the form [guaranteed to work] of:

$$y_2 = y_1 \ln x + c_0^* x^r + c_1^* x^{r+1} + c_2^* x^{r+2} + ...$$

where y_1 is the first basic solution (with c_0 set equal to 1, i.e. removing the multiplicative constant). The corresponding recurrence formula (for the c_i^*'s) will offer us a free choice of c_0^*, which we normally set equal to 0 (a nonzero choice would only add $c_0^* y_1$ to our second basic solution). After that, the rest of the c_i^*'s uniquely follows (they may turn out to be all 0 in some cases).

EXAMPLES:

- $(1+x)x^2y'' - (1+2x)xy' + (1+2x)y = 0$ [$a(x) = -\frac{1+2x}{1+x}$ and $b(x) = \frac{1+2x}{1+x}$]. The indicial equation is $r^2 - 2r + 1 = 0 \Rightarrow r_{1,2} = 1 \pm 0$ [double]. Substituting $\sum\limits_{i=0}^{\infty} c_i x^{i+1}$ for y yields $\sum\limits_{i=0}^{\infty} c_i(i+1)ix^{i+1} + \sum\limits_{i=0}^{\infty} c_i(i+1)ix^{i+2} - \sum\limits_{i=0}^{\infty} c_i(i+1)x^{i+1} - 2\sum\limits_{i=0}^{\infty} c_i(i+1)x^{i+2} + \sum\limits_{i=0}^{\infty} c_i x^{i+1} + 2\sum\limits_{i=0}^{\infty} c_i x^{i+2} = 0$. Combining terms with like powers of x: $\sum\limits_{i=0}^{\infty} c_i i^2 x^{i+1} + \sum\limits_{i=0}^{\infty} c_i i(i-1)x^{i+2} = 0$. Adjusting the index of the second sum: $\sum\limits_{i=0}^{\infty} c_i i^2 x^{i+1} + \sum\limits_{i=1}^{\infty} c_{i-1}(i-1)(i-2)x^{i+1} = 0$. The 'exceptional' $i = 0$ term must equal to zero automatically, our indicial equation takes care of that [check], the rest implies $c_i = -\frac{(i-1)(i-2)}{i^2}c_{i-1}$ for $i = 1, 2, 3, ...$, yielding $c_1 = 0$, $c_2 = 0$,.... The first basic solution is thus $c_0 x$ [i.e. $y_1 = x$, verify!]. Once we have identified the first basic solution as a simple function [when lucky] we have *two* options:

1. (a) Use V of P: $y(x) = c(x)\cdot x \Rightarrow (1+x)xc'' + c' = 0 \Rightarrow \frac{dz}{z} = (\frac{1}{1+x} - \frac{1}{x})\,dx \Rightarrow \ln z = \ln(1+x) - \ln x + \tilde{c} \Rightarrow c' = c_0^* \frac{1+x}{x} \Rightarrow c(x) = c_0^*(\ln x + x) + c_0$. This makes it clear that the second basic solution is $x\ln x + x^2$.

   (b) Insist on using Frobenius: Substitute $y^{(T)} = x\ln x + \sum\limits_{i=0}^{\infty} c_i^* x^{i+1}$ into the original equation. The sum will give you the same contribution as before, the $x \ln x$ term (having no unknowns) yields an extra, non-homogeneous term of the corresponding recurrence equation. There is a bit of an automatic simplification when substituting $y_1 \ln x$ (our $x \ln x$) into the equation, as the $\ln x$-proportional terms must cancel. What we need is thus $y \to 0$, $y' \to \frac{y_1}{x}$ and $y'' \to 2\frac{y_1'}{x} - \frac{y_1}{x^2}$ .This substitution results in the same old [except for $c \to c^*$] $\sum_{i=0}^{\infty} c_i^* i^2 x^{i+1} + \sum_{i=1}^{\infty} c_{i-1}^*(i-1)(i-2)x^{i+1}$ on the left hand side of the equation, and $-(1+x)x^2 \cdot \frac{1}{x} + (1+2x)x = x^2$ [don't forget to reverse the sign] on the right hand side. This yields the same set of recurrence formulas as before, *except* at $i = 1$ [due to the nonzero right-hand-side term]. Again we get a 'free choice' of c_0^* [indicial equation takes care of that], which we utilize by setting c_0^* equal to *zero* (or anything which simplifies the answer), since a nonzero c_0^* would only add a redundant $c_0^* y_1$ to our *second* basic solution. The x^2-part of the equation ($i = 1$) then reads: $c_1^* x^2 + 0 = x^2 \Rightarrow c_1^* = 1$. The rest of the sequence follows from $c_i^* = -\frac{(i-1)(i-2)}{i^2} c_{i-1}^*$, $i = 2, 3, 4...$ $\Rightarrow c_2^* = c_3^* = ... = 0$ as before. The second basic solution is thus $y_1 \ln x + c_1^* x^2 = x \ln x + x^2$ [check].

- $x(x-1)y'' + (3x-1)y' + y = 0$ $[a(x) = \frac{3x-1}{x-1}$ $b(x) = \frac{x}{x-1}] \Rightarrow r^2 = 0$ [double root of 0]. Substituting $y^{(T)} = \sum_{i=0}^{\infty} c_i x^{i+0}$ yields $\sum_{i=0}^{\infty} i(i-1)c_i x^i - \sum_{i=0}^{\infty} i(i-1)c_i x^{i-1} + 3\sum_{i=0}^{\infty} i c_i x^i - \sum_{i=0}^{\infty} i c_i x^{i-1} + \sum_{i=0}^{\infty} c_i x^i = 0 \Leftrightarrow \sum_{i=0}^{\infty}[i^2+2i+1]c_i x^i - \sum_{i=0}^{\infty} i^2 c_i x^{i-1} = 0 \Leftrightarrow$ $\sum_{i=0}^{\infty}(i+1)^2 c_i x^i - \sum_{i=-1}^{\infty}(i+1)^2 c_{i+1} x^i = 0$. The lowest, $i = -1$ coefficient is zero automatically, thus c_0 is arbitrary. The remaining coefficients are $(i+1)^2[c_i - c_{i+1}]$, set to zero $\Rightarrow c_{i+1} = c_i$ for $i = 0, 1, 2,.... \Rightarrow c_0 = c_1 = c_2 = c_3 = ... \Rightarrow 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$ is the first basic solution. Again, we can get the second basic solution by either the V of P or Frobenius technique. We demonstrate only the latter: $y^{(T)} = \frac{\ln x}{1-x} + \sum_{i=0}^{\infty} c_i^* x^{i+0}$ getting the same left hand side and the following right hand side: $x(x-1)\left[\frac{2}{x(1-x)^2} - \frac{1}{x^2(1-x)}\right] + (3x-1) \cdot \frac{1}{x(1-x)} = 0$ [not typical, but it may happen]. This means that not only c_0^*, but all the other c^*-coefficients can be set equal to zero. The second basic solution is thus $\frac{\ln x}{1-x}$ [which can be verified easily by direct substitution].

► $r_1 - r_2$ Equals a Positive Integer◄

(we *choose* $r_1 > r_2$).

The first basic solution can be constructed, based on $y^{(T)} = \sum_{i=0}^{\infty} c_i x^{i+r_1}$, in the usual manner (don't forget that r_1 should be the *bigger* root). The second basic solution will then have the form of

$$K y_1 \ln x + \sum_{i=0}^{\infty} c_i^* x^{i+r_2}$$

where K becomes one of the *unknowns* (on par with the c_i^*'s), but it may turn out to have a zero value. Note that we will first have a free choice of c_0^* (must be non-zero) and then, when we reach it, we will also be offered a free choice of $c_{r_1-r_2}^*$ (to simplify the solution, we usually set it equal to zero – a nonzero choice would only add an extra multiple of y_1).

EXAMPLES:

- $(x^2-1)x^2y''-(x^2+1)xy'+(x^2+1)y=0$ $[a(x)=-\frac{x^2+1}{x^2-1}$ and $b(x)=\frac{x^2+1}{x^2-1}]\Rightarrow$ $r^2-1=0\Rightarrow r_{1,2}=1$ and -1. Using $y^{(T)}=\sum_{i=0}^{\infty}c_ix^{i+1}$ we get: $\sum_{i=0}^{\infty}(i+1)ic_ix^{i+3}-\sum_{i=0}^{\infty}(i+1)ic_ix^{i+1}-\sum_{i=0}^{\infty}(i+1)c_ix^{i+3}-\sum_{i=0}^{\infty}c_i(i+1)x^{i+1}+\sum_{i=0}^{\infty}c_ix^{i+3}+\sum_{i=0}^{\infty}c_ix^{i+1}=0\Leftrightarrow\sum_{i=0}^{\infty}i^2c_ix^{i+3}-\sum_{i=0}^{\infty}i(i+2)c_ix^{i+1}=0\Leftrightarrow\sum_{i=0}^{\infty}i^2c_ix^{i+3}-\sum_{i=-2}^{\infty}(i+2)(i+4)c_{i+2}x^{i+3}=0$. The lowest $i=-2$ term is zero automatically [$\Rightarrow c_0$ can have any value], the next $i=-1$ term [still 'exceptional'] disappears only when $c_1=0$. The rest of the c-sequence follows from $c_{i+2}=\frac{i^2c_i}{(i+2)(i+4)}$ with $i=0,\ 1,\ 2,...\Rightarrow c_2=c_3=c_4=....=0$. The first basic solution is thus c_0x [$y_1=x$, discarding the constant]. To construct the second basic solution, we substitute $Kx\ln x+\sum_{i=0}^{\infty}c_i^*x^{i-1}$ for y, getting: $\sum_{i=0}^{\infty}(i-1)(i-2)c_ix^{i+1}-\sum_{i=0}^{\infty}(i-1)(i-2)c_ix^{i-1}-\sum_{i=0}^{\infty}(i-1)c_ix^{i+1}-\sum_{i=0}^{\infty}c(i-1)_ix^{i-1}+\sum_{i=0}^{\infty}c_ix^{i+1}+\sum_{i=0}^{\infty}c_ix^{i-1}=\sum_{i=0}^{\infty}(i-2)^2c_ix^{i+1}-\sum_{i=0}^{\infty}i(i-2)c_ix^{i-1}=\sum_{i=0}^{\infty}(i-2)^2c_ix^{i+1}-\sum_{i=-2}^{\infty}(i+2)ic_{i+2}x^{i+1}$ on the left hand side, and $-(x^2-1)x^2\cdot\frac{K}{x}+(x^2+1)x\cdot K=2Kx$ on the right hand side (the contribution of $Kx\ln x$). The $i=-2$ term allows c_0^* to be arbitrary, $i=-1$ requires $c_1^*=0$, and $i=0$ [due to the right hand side, the x^1-terms must be also considered 'exceptional'] requires $4c_0^*=2K\Rightarrow$ $K=2c_0^*$, and leaves c_2^* free for us to choose (we take $c_2^*=0$). After that, $c_{i+2}^*=\frac{(i-2)^2}{(i+2)i}c_i^*$ where $i=1,\ 2,\ 3,\ ....\Rightarrow c_3^*=c_4^*=c_5^*=.....=0$. The second basic solution is thus $c_0^*\left(2x\ln x+\frac{1}{x}\right)$ [verify!].

- $x^2y''+xy'+(x^2-\frac{1}{4})y=0\Rightarrow r^2-\frac{1}{4}=0\Rightarrow r_{1,2}=\frac{1}{2}$ and $-\frac{1}{2}$. Substituting $y^{(T)}=\sum_{i=0}^{\infty}c_ix^{i+1/2}$, we get $\sum_{i=0}^{\infty}(i+\frac{1}{2})(i-\frac{1}{2})c_ix^{i+1/2}+\sum_{i=0}^{\infty}(i+\frac{1}{2})c_ix^{i+1/2}+\sum_{i=0}^{\infty}c_ix^{i+5/2}-\frac{1}{4}\sum_{i=0}^{\infty}c_ix^{i+1/2}=0\Leftrightarrow\sum_{i=0}^{\infty}(i+1)ic_ix^{i+1/2}+\sum_{i=0}^{\infty}c_ix^{i+5/2}=0\Leftrightarrow\sum_{i=-2}^{\infty}(i+3)(i+2)c_{i+2}x^{i+5/2}+\sum_{i=0}^{\infty}c_ix^{i+5/2}=0$, which yields: a free choice of c_0, $c_1=0$ and $c_{i+2}=-\frac{c_i}{(i+2)(i+3)}$ where $i=0,\ 1,.....\Rightarrow c_2=-\frac{c_0}{3!}$, $c_4=\frac{c_0}{5!}$, $c_6=-\frac{c_0}{7!}$, ..., and $c_3=c_5=...=0$. The first basic solution thus equals $c_0x^{1/2}(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+...)=c_0\frac{\sin x}{\sqrt{x}}$ [$\Rightarrow y_1=\frac{\sin x}{\sqrt{x}}$]. Substituting $Ky_1\ln x+\sum_{i=0}^{\infty}c_i^*x^{i-1/2}$ for y similarly reduces the equation to $\sum_{i=0}^{\infty}(i-1)ic_i^*x^{i-1/2}+\sum_{i=0}^{\infty}c_i^*x^{i+3/2}$ on the left hand side and $-x^2\cdot(-\frac{y_1}{x^2}+\frac{2}{x}y_1')-x\cdot\frac{y_1}{x} = -2xy_1' = K(-x^{1/2}+\frac{5x^{5/2}}{3!}-\frac{9x^{9/2}}{5!}+...)$ on the right hand side or, equivalently, $\sum_{i=-2}^{\infty}(i+1)(i+2)c_{i+2}^{*}x^{i+3/2}+\sum_{i=0}^{\infty}c_i^{*}x^{i+3/2} = K(-x^{1/2}+\frac{5x^{5/2}}{3!}-\frac{9x^{9/2}}{5!}+...)$. This implies that c_0^* can have any value ($i=-2$), c_1^* can also have any value (we *make* it 0), K *must* equal zero ($i=-1$), and $c_{i+2}^* = -\frac{c_i^*}{(i+1)(i+2)}$ for $i = 0,\ 1,\ 2,\ ...$ $\Rightarrow c_2^* = -\frac{c_0^*}{2!},\ c_4^* = \frac{c_0^*}{4!},\ c_6^* = -\frac{c_0^*}{6!},\ ...\ \Rightarrow$ $y_2 = x^{-1/2}(1-\frac{x^2}{2!}+\frac{x^4}{4!}-....) = \frac{\cos x}{\sqrt{x}}$. ■

In each of the previous examples the second basic solution could have been constructed by V of P – try it.

Also note that so far we have avoided solving a truly non-homogeneous recurrence formula – K never appeared in more than one of its (infinitely many) equations.

A few Special functions of Mathematical Physics

This section demonstrates various applications of the Frobenius technique.

►Laguerre Equation◄

$$y''+\frac{1-x}{x}y'+\frac{n}{x}y=0$$

or, equivalently:

$$(xe^{-x}y')'+nye^{-x}=0$$

which identifies it as an eigenvalue problem, with the solutions being orthogonal in the $\int_0^\infty e^{-x}L_{n_1}(x)\cdot L_{n_2}(x)\,dx$ sense.

Since $a(x)=1-x$ and $b(x)=x$ we get $r^2=0$ [duplicate roots]. Substituting $\sum_{i=0}^{\infty}c_ix^i$ for y in the original equation (multiplied by x) results in $\sum_{i=0}^{\infty}i^2c_ix^{i-2}+\sum_{i=0}^{\infty}(n-i)c_ix^{i-1}=0 \Leftrightarrow \sum_{i=-1}^{\infty}(i+1)^2c_{i+1}x^{i-1}+\sum_{i=0}^{\infty}(n-i)c_ix^{i-1}=0 \Rightarrow c_{i+1}=-\frac{n-i}{(i+1)^2}c_i$ for $i=0,\ 1,\ 2,\ ....$ Only polynomial solutions are square integrable in the above sense (relevant to Physics), so n must be an *integer*, to make c_{n+1} and all subsequent c_i-values equal to 0 and thus solve the eigenvalue problem.

The first basic solution is thus $L_n(x)$ [the standard notation for **Laguerre polynomials**] =

$$1-\frac{n}{1^2}x+\frac{n(n-1)}{(2!)^2}x^2-\frac{n(n-1)(n-2)}{(3!)^2}x^3+....\pm\frac{1}{n!}x^n$$

The second basic solution does *not* solve the eigenvalue problem (it is not square integrable), so we will not bother to construct it [not that it should be difficult – try it if you like].

Optional: Based on the Laguerre polynomials, one can develop the following solution to one of the most important problems in Physics (Quantum-Mechanical treatment of Hydrogen atom):

We know that $xL''_{n+m} + (1-x)L'_{n+m} + (n+m)L_{n+m} = 0$ [n and m are two integers; this is just a restatement of the Laguerre equation]. Differentiating $2m+1$ times results in $xL^{(2m+3)}_{n+m} + (2m+1)L^{(2m+2)}_{n+m} + (1-x)L^{(2m+2)}_{n+m} - (2m+1)L^{(2m+1)}_{n+m} + (n+m)L^{(2m+1)}_{n+m} = 0$, clearly indicating that $L^{(2m+1)}_{n+m}$ is a solution to $xy'' + (2m+2-x)y' + (n+m-1)y = 0$. Introducing a new *dependent* variable $u(x) = x^{m+1}e^{-x/2}y$, i.e. substituting $y = x^{-m-1}e^{x/2}u$ into the previous equation, leads to $u'' + [-\frac{1}{4} + \frac{n}{x} - \frac{m(m+1)}{x^2}]u = 0$. Introducing a new *independent* variable $z = \frac{n}{2}x$ [$\Rightarrow u' = \frac{n}{2}\dot{u}$ and $u'' = (\frac{n}{2})^2\ddot{u}$ where each dot implies a z-derivative] results in $\ddot{u} + \left[-\frac{1}{n^2} + \frac{2}{z} - \frac{m(m+1)}{z^2}\right]u = 0$.

We have thus effectively solved the following (S-L) eigenvalue problem:

$$\ddot{u} + \left[\lambda + \frac{2}{z} - \frac{m(m+1)}{z^2}\right]u = 0$$

[m considered fixed], proving that the eigenvalues are $\lambda = -\frac{1}{n^2}$ and constructing the respective eigenfunctions [the so called ORBITALS]: $u(z) = (\frac{2z}{n})^{m+1}e^{-z/2}L^{(2m+1)}_{n+m}(\frac{2z}{n})$. Any two such functions with the same m but distinct n_1 and n_2 will be orthogonal, thus: $\int\limits_0^\infty u_1(z)u_2(z)\,dz = 0$ [recall the general L-S theory relating to $(pu')' + (\lambda q + r)u = 0$]. Understanding this short example takes care of a nontrivial chunk of modern Physics. ⊗

►Bessel equation◄

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

where n has any (non-negative) value.

The **indicial equation** is $r^2 - n^2 = 0$ yielding $r_{1,2} = n, -n$.

To build the **first basic solution** we use $y^{(T)} = \sum\limits_{i=0}^\infty c_i x^{i+n} \Rightarrow \sum\limits_{i=0}^\infty i(i+2n)c_i x^{i+n} + \sum\limits_{i=0}^\infty c_i x^{i+n+2} = 0 \Leftrightarrow \sum\limits_{i=0}^\infty i(i+2n)c_i x^{i+n} + \sum\limits_{i=2}^\infty c_{i-2}x^{i+n} = 0 \Rightarrow c_0$ arbitrary, $c_1 = c_3 = c_5 = \ldots = 0$ and $c_i = -\frac{c_{i-2}}{i(2n+i)}$ for $i = 2, 4, 6, \ldots \Rightarrow c_2 = -\frac{c_0}{2(2n+2)}$, $c_4 = \frac{c_0}{4\times2\times(2n+2)\times(2n+4)}$, $c_6 = -\frac{c_0}{6\times4\times2\times(2n+2)\times(2n+4)\times(2n+6)}$, ..., $c_{2k} = \frac{(-1)^k c_0}{2^{2k}(n+1)(n+2).....(n+k)k!}$ in general, where $k = 0, 1, 2, \ldots$. When n is an *integer*, the last expression can be written as $c_{2k} = \frac{(-1)^k n! c_0}{2^{2k}(n+k)!k!} \equiv \frac{(-1)^k \tilde{c}_o}{2^{2k+n}\,k!\,(n+k)!}$. The first basic solution is thus

$$\sum_{k=0}^\infty \frac{(-1)^k(\frac{x}{2})^{2k+n}}{k!(n+k)!}$$

It is called the BESSEL FUNCTION of the *first kind* of 'order' n [note that the 'order' has nothing to do with the order of the corresponding equation, which is always 2], the standard *notation* being $J_n(x)$; its values (if not on your calculator) can be found in tables.

When n is a **non-integer**, one has to extend the definition of the factorial function to non-integer arguments. This extension is called a Γ-FUNCTION, and is 'shifted' with respect to the factorial function, thus: $n! \equiv \Gamma(n+1)$. For positive α $(= n+1)$ values, it is achieved by the following integral

$$\Gamma(\alpha) \equiv \int_0^\infty x^{\alpha-1} e^{-x} dx$$

[note that for integer α this yields $(\alpha-1)!$], for negative α values the extension is done with the help of

$$\Gamma(\alpha-1) = \frac{\Gamma(\alpha)}{\alpha-1}$$

[its values can often be found on your calculator].

Using this extension, the previous $J_n(x)$ solution (of the Bessel equation) becomes correct for any n [upon the $(n+k)! \to \Gamma(n+k+1)$ replacement].

When n is *not* an integer, the same formula with $n \to -n$ provides **the second basic solution** [easy to verify].

Of the non-integer cases, the most important are those with a **half-integer** value of n. One can easily verify [you will need $\Gamma(\frac{1}{2}) = \sqrt{\pi}$] that the corresponding Bessel functions are elementary, e.g.

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \sin x \\ J_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \cos x \\ J_{\frac{3}{2}}(x) &= \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) \\ &\cdots \end{aligned}$$

Unfortunately, the most common is the case of n being an **integer.**

Constructing the **second basic solution** is then a lot more difficult. It has, as we know, the form of $K y_1 \ln x + \sum_{i=0}^{\infty} c_i^* x^{i-n}$. Substituting this into the Bessel equation yields $\sum_{i=0}^{\infty} i(i-2n) c_i^* x^{i-n} + \sum_{i=2}^{\infty} c_{i-2}^* x^{i-n}$ on the left hand side and $-K \left[x^2 \cdot (2\frac{y_1'}{x} - \frac{y_1}{x^2}) + x \cdot \frac{y_1}{x} \right] = -2K \sum_{k=0}^{\infty} \frac{(-1)^k (2k+n) (\frac{x}{2})^{2k+n}}{k!(n+k)!} \equiv$ $-2K \sum_{k=n}^{\infty} \frac{(-1)^{k-n} (2k-n) (\frac{x}{2})^{2k-n}}{(k-n)!k!}$ on the right hand side of the recurrence formula.

One can solve it by taking c_0^* to be arbitrary, $c_1^* = c_3^* = c_5^* = ... = 0$, and $c_2^* = \frac{c_0^*}{2(2n-2)}$, $c_4^* = \frac{c_0^*}{4\times2\times(2n-2)\times(2n-4)}$, $c_6^* = \frac{c_0^*}{6\times4\times2\times(2n-2)\times(2n-4)\times(2n-6)}$, ...,

$$c_{2k}^* = \frac{c_0^*}{2^{2k}(n-1)(n-2).....(n-k)k!} \equiv \frac{c_0^*(n-k-1)!}{2^{2k}(n-1)!k!}$$

up to and including $k = n-1$ $[i = 2n-2]$. When we reach $i = 2n$ the right hand side starts contributing! The overall coefficient of x^n is $c_{2n-2}^* = -2K \frac{1}{2^n\,(n-1)!} \Rightarrow$

$$K = \frac{-c_0^*}{2^{n-1}(n-1)!}$$

allowing a free choice of c^*_{2n}.

To solve the remaining part of the recurrence formula (truly non-homogeneous) is more difficult, so we only quote (and verify) the answer:

$$c^*_{2k} = c^*_0 \frac{(-1)^{k-n}(h_{k-n} + h_k)}{2^{2k}(k-n)!k!(n-1)!}$$

for $k \geq n$, where $h_k = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k}$.

Proof: Substituting this $c^*_{2k} \equiv c^*_i$ into the recurrence formula and cancelling the common part of $\frac{(-1)^{k-n}\, c^*_0}{2^{2k}\,(k-n)!k!(n-1)!} x^{2k-n}$ yields: $2k(2k-2n)(h_{k-n}+h_k) - 4k(k-n)(h_{k-n-1} + h_{k-1}) = 4(2k-n)$. This is a true identity as $h_{k-n} - h_{k-n-1} + h_k - h_{k-1} = \frac{1}{k-n} + \frac{1}{k} = \frac{2k-n}{k(k-n)}$ [multiply by $4k(k-n)$]. □

The second basic solution is usually written (a slightly different normalizing constant is used, and a bit of $J_n(x)$ is added) as:

$$\begin{aligned} Y_n(x) &= \frac{2}{\pi} J_n(x) \left[\ln \frac{x}{2} + \gamma \right] + \frac{1}{\pi} \sum_{k=n}^{\infty} \frac{(-1)^{k+1-n}(h_{k-n} + h_k)}{(k-n)!k!} \left(\frac{x}{2} \right)^{2k-n} \\ &\quad - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2} \right)^{2k-n} \end{aligned}$$

where γ is the Euler constant ≈ 0.557 [the reason for the extra term is that the last formula is derived based on yet another, possibly more elegant approach than ours, namely: $\lim\limits_{\upsilon \to n} \frac{J_\nu \cos(\nu\pi) - J_{-\nu}}{\sin(\nu\pi)}$]. $Y_n(x)$ is called the Bessel function of SECOND KIND of order n.

More on **Bessel functions:**

To deal with initial-value and boundary-value problems, we have to be able to evaluate J_n, J'_n, Y_n and Y'_n [concentrating on integer n]. The tables on page A97 of your textbook provide only J_0 and J_1, the rest can be obtained by repeated application of

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

and/or

$$J'_n(x) = \frac{J_{n-1}(x) - J_{n+1}(x)}{2}$$

[with the understanding that $J_{-1}(x) = -J_1(x)$ and $\lim\limits_{x \to 0} \frac{J_n(x)}{x} = 0$ for $n = 1, 2, 3, ...$], and the same set of formulas with Y in place of J.

Proof: $[(\frac{x}{2})^n J_n]' = \frac{n}{2}(\frac{x}{2})^{n-1} J_n + (\frac{x}{2})^n J'_n = \sum\limits_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+2n-1}}{k!(n+k-1)!} = (\frac{x}{2})^n J_{n-1}$ and $[(\frac{x}{2})^{-n} J_n]' = -\frac{n}{2}(\frac{x}{2})^{-n-1} J_n + (\frac{x}{2})^{-n} J'_n = \sum\limits_{k=1}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k-1}}{(k-1)!(n+k)!} = \sum\limits_{k=0}^{\infty} \frac{(-1)^{k+1} (\frac{x}{2})^{2k+1}}{k!(n+k+1)!} - (\frac{x}{2})^{-n} J_{n+1}$. Divided by $(\frac{x}{2})^n$ and $(\frac{x}{2})^{-n}$ respectively, these give

$$\frac{n}{x} J_n + J'_n = J_{n-1}$$

and

$$-\frac{n}{x}J_n + J_n' = -J_{n+1}$$

Adding and subtracting the two yields the rest [for the Y functions the proof would slightly more complicated, but the results are the same]. □

EXAMPLES:

1. $J_3(1.3) = \frac{4}{1.3}J_2(1.3) - J_1(1.3) = \frac{4}{1.3}[\frac{2}{1.3}J_1(1.3) - J_0(1.3)] - J_1(1.3) = \frac{4}{1.3}[\frac{2}{1.3} \times 0.52202 - 0.62009] - 0.52202 = 0.0411$

2. $J_2'(1.3) = \frac{1}{2}[J_1(1.3) - J_3(1.3)] = \frac{1}{2}[0.52202 - 0.0411] = 0.2405$ ■

Modified Bessel equation:

$$x^2y'' + xy' - (x^2 + n^2)y = 0$$

[differs from Bessel equation by a single sign]. The two basic solutions can be developed in almost an identical manner to the 'unmodified' Bessel case [the results differ only be an occasional sign]. We will not duplicate our effort, and only mention the new notation: the two basic solutions are now $I_n(x)$ and $K_n(x)$ [MODIFIED BESSEL FUNCTIONS of first and second kind]. Only I_0 and I_1 need to be tabulated as $I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x}I_n$ and $I_n' = \frac{I_{n-1}+I_{n+1}}{2}$ (same with $I_n \to K_n$).

Transformed Bessel equation:

$$x^2y'' + (1-2a)xy' + (b^2c^2x^{2c} - n^2c^2 + a^2)y = 0$$

where a, b, c and n are arbitrary constants [the equation could have been written as $x^2y'' + Axy' + (B^2x^C - D)y = 0$, but the above parametrization is more convenient].

To find the solution we substitute $y(x) = x^a \cdot u(x)$ [introducing new *dependent* variable u] getting: $a(a-1)u + 2axu' + x^2u'' + (1-2a)(au + xu') + (b^2c^2x^{2c} - n^2c^2 + a^2)u =$

$$x^2u'' + xu' + (b^2c^2x^{2c} - n^2c^2)u = 0$$

Then we introduce $z = bx^c$ as a new *independent* variable [recall that $u' \to \frac{du}{dz} \cdot bcx^{c-1}$ and $u'' \to \frac{d^2u}{dz^2} \cdot (bcx^{c-1})^2 + \frac{du}{dz} \cdot bc(c-1)x^{c-2}$] $\Rightarrow x^2 \cdot \left(\frac{d^2u}{dz^2} \cdot (bcx^{c-1})^2 + \frac{du}{dz} \cdot bc(c-1)x^{c-2}\right) + x \cdot \left(\frac{du}{dz} \cdot bcx^{c-1}\right) + (b^2c^2x^{2c} - n^2c^2)u =$ [after cancelling c^2]

$$z^2 \cdot \frac{d^2u}{dz^2} + z \cdot \frac{du}{dz} + \left(z^2 - n^2\right)u = 0$$

which is the Bessel equation, having $u(z) = C_1J_n(z) + C_2Y_n(z)$ [or $C_2J_{-n}(x)$ when n is *not* an integer] as its general solution.

The **solution** to the original equation is thus

$$C_1x^aJ_n(bx^c) + C_2x^aY_n(bx^c)$$

EXAMPLES:

1. $xy'' - y' + xy = 0$ [same as $x^2y'' - xy' + x^2y = 0$] $\Rightarrow a = 1$ [from $1 - 2a = -1$], $c = 1$ [from $b^2c^2x^{2c}y = x^2y$], $b = 1$ [from $b^2c^2 = 1$] and $n = 1$ [from $a^2 - n^2c^2 = 0$] $\Rightarrow$
$$y(x) = C_1 x J_1(x) + C_2 x Y_1(x)$$

2. $x^2y'' - 3xy' + 4(x^4 - 3)y = 0 \Rightarrow a = 2$ [from $1 - 2a = -3$], $c = 2$ [from $b^2c^2x^{2c}y = 4x^4y$], $b = 1$ [from $b^2c^2 = 4$] and $n = 2$ [from $a^2 - n^2c^2 = -12$] $\Rightarrow$
$$y = C_1 x^2 J_2(x^2) + C_2 x^2 Y_2(x^2)$$

3. $x^2y'' + (\frac{81}{4}x^3 - \frac{35}{4})y = 0 \Rightarrow a = \frac{1}{2}$ [from $1 - 2a = 0$], $c = \frac{3}{2}$ [from x^3], $b = 3$ [from $b^2c^2 = \frac{81}{4}$] and $n = 2$ [from $a^2 - n^2c^2 = -\frac{35}{4}$] $\Rightarrow$
$$y = C_1\sqrt{x} J_2(3x^{3/2}) + C_2\sqrt{x} Y_2(3x^{3/2})$$

4. $x^2y'' - 5xy' + (x + \frac{35}{4})y = 0 \Rightarrow a = 3$ [$1 - 2a = -5$], $c = \frac{1}{2}$ [xy], $b = 2$ [$b^2c^2 = 1$] and $n = 1$ [$a^2 - n^2c^2 = \frac{35}{4}$] $\Rightarrow$
$$y = C_1 x^3 J_1(2\sqrt{x}) + C_2 x^3 Y_1(2\sqrt{x})$$

▶Hypergeometric equation◀

$$x(1 - x)y'' + [c - (a + b + 1)x]y' - aby = 0$$

$\Rightarrow r^2 + r(c - 1) = 0 \Rightarrow r_{1,2} = 0$ and $1 - c$.

Substituting $y^{(T)} = \sum_{i=0}^{\infty} c_i x^i$ yields: $\sum_{i=-1}^{\infty} (i+1)(i+c)c_{i+1}x^i - \sum_{i=0}^{\infty}(i+a)(i+b)c_i x^i \Rightarrow$ $c_1 = \frac{ab}{1\cdot c}c_0$, $c_2 = \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}c_0$, $c_3 = \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}c_0$, ... which shows that the **first basic solution** is

$$1 + \frac{ab}{1\cdot c}x + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}x^3 + ...$$

The usual **notation** for this series is $F(a, b; c; x)$, and it is called the HYPERGEOMETRIC FUNCTION. Note that a and b are interchangeable.. Also note that when either of them is a negative integer (or zero), $F(a, b; c; x)$ is just a simple polynomial (of the corresponding degree) – please learn to identify it as such!

Similarly, when c is *noninteger* [to avoid Case 3], we can show [skipping the details now] that the **second basic solution** is

$$x^{1-c}F(a + 1 - c, b + 1 - c; 2 - c; x)$$

[this may be correct even in some Case 3 situations, but don't forget to verify it].

EXAMPLE:

1. $x(1-x)y'' + (3-5x)y' - 4y = 0 \Rightarrow ab = 4$, $a+b+1 = 5 \Rightarrow b^2 - 4b + 4 = 0 \Rightarrow$ $a = 2$, $b = 2$, and $c = 3 \Rightarrow C_1F(2, 2; 3; x) + C_2x^{-2}F(0, 0, ; -1; x)$ [the second part is subject to verification]. Since $F(0, 0; -1; x) \equiv 1$, the second basic solution is x^{-2}, which *does meet* the equation [substitute].

Transformed Hypergeometric equation:

$$(x-x_1)(x_2-x)y''+[D-(a+b+1)x]y'-aby=0$$

where x_1 and x_2 (in addition to a, b, and D) are specific *numbers*.

One can easily verify that changing the *independent* variable to $z=\dfrac{x-x_1}{x_2-x_1}$ transforms the equation to

$$z(1-z)\frac{d^2y}{dz^2}+\left[\frac{D-(a+b+1)x_1}{x_2-x_1}-(a+b+1)z\right]\frac{dy}{dz}-aby=0$$

which we know how to solve [hypergeometric].

EXAMPLES:

1. $4(x^2-3x+2)y''-2y'+y=0\Rightarrow(x-1)(2-x)y''+\frac{1}{2}y'-\frac{1}{4}y=0\Rightarrow x_1=1$, $x_2=2$, $ab=\frac{1}{4}$ and $a+b+1=0\Rightarrow b^2+b+\frac{1}{4}=0\Rightarrow a=-\frac{1}{2}$ and $b=-\frac{1}{2}$, and finally $c=\frac{\frac{1}{2}-(a+b+1)x_1}{x_2-x1}=\frac{1}{2}$. The solution is thus

$$y=C_1F(-\frac{1}{2},-\frac{1}{2};\frac{1}{2};x-1)+C_2(x-1)^{1/2}F(0,0;\frac{3}{2};x-1)$$

[since $z=x-1$]. Note that $F(0,0;\frac{3}{2};x-1)\equiv1$ [*some* hypergeometric functions are elementary or even trivial, e.g. $F(1,1;2;x)\equiv-\frac{\ln(1-x)}{x}$, etc.].

2. $3x(1+x)y''+xy'-y=0\Rightarrow(x+1)\,(0-x)\,y''-\frac{1}{3}xy+\frac{1}{3}y=0\Rightarrow x_1=-1$ [note the sign!] $x_2=0$, $ab=-\frac{1}{3}$ and $a+b+1=\frac{1}{3}\Rightarrow a=\frac{1}{3}$ and $b=-1\Rightarrow$ $c=\frac{0-\frac{1}{3}(-1)}{1}=\frac{1}{3}\Rightarrow$

$$y=C_1F(\frac{1}{3},-1;\frac{1}{3};x+1)+C_2(x+1)^{2/3}F(1,-\frac{1}{3};\frac{5}{3};x+1)$$

[the first $F(...)$ equals to $-x$; coincidentally, even the second $F(...)$ can be converted to a rather lengthy expression involving ordinary functions]. ■

Part II

VECTOR ANALYSIS

Chapter 7 FUNCTIONS IN THREE DIMENSIONS – DIFFERENTIATION

3-D Geometry (overview)

It was already agreed (see Prerequisites) that everyone understands the concept of Cartesian (right-handed) coordinates, and is able to visualize points and (free) vectors within this framework. Don't forget that both vectors and point are represented by a triplet on numbers, e.g. (2,1,-4). For their names, I will normally use small boldface letters (e.g. **a**, **b**, **c**, ... in these notes, but $\vec{a}$, $\vec{b}$, $\vec{c}$ on the board).

▷ Notation and terminology

$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$ is the vector's LENGTH or MAGNITUDE (a_x, a_y and a_z are the vector's three components; sometimes I may also call them a_1, a_2 and a_3).

When $|\mathbf{u}| = 1$, $\mathbf{u}$ is called a UNIT vector (representing a DIRECTION). $\mathbf{e}_1$ ($\mathbf{e}_2$, $\mathbf{e}_3$) is the unit vector of the $+x$ ($+y$, $+z$) direction, respectively.

$(0, 0, 0)$ is called a ZERO vector.

Multiplying every component of a vector by the same SCALAR (single) number is called SCALAR MULTIPLICATION, e.g. $3 \cdot (2, -1, 4) = (6, -3, 12)$. Geometrically, this represents modifying the vector's length according to the scalar's magnitude, without changing direction [a negative value of the scalar also changes the vector's ORIENTATION].

ADDITION of two vectors is the corresponding component-wise operation, e.g.: $(3, -1, 2) + (4, 0, -3) = (7, -1, -1)$. It is clearly commutative, i.e. $\mathbf{a} + \mathbf{b} \equiv \mathbf{b} + \mathbf{a}$ [be able to visualize this].

▷ **Dot** (inner) [scalar] **product**

of two vectors is *defined* by

$$\mathbf{a} \bullet \mathbf{b} \equiv |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\gamma$$

(a *scalar* result), where γ is the angle between the direction of **a** and **b** [anywhere from 0 to π]. Geometrically, this corresponds to the length of the *projection* of **a** into the direction of **b**, multiplied by $|\mathbf{b}|$ (or, equivalently, reverse). It is usually *computed* based on

$$\mathbf{a} \bullet \mathbf{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$$

[e.g. $(2, -3, 1) \bullet (4, 2, -3) = 8 - 6 - 3 = -1$], and it is obviously *commutative* [i.e. $\mathbf{a} \bullet \mathbf{b} \equiv \mathbf{b} \bullet \mathbf{a}$].

To prove the equivalence of the two definitions, your textbook starts with $|\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\gamma$ and reduces it to $a_1 b_1 + a_2 b_2 + a_3 b_3$. The crucial part of their proof is the following *distributive law*, which they don't justify: $(\mathbf{a} + \mathbf{b}) \bullet \mathbf{c} \equiv \mathbf{a} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c}$. To see why it is correct, think of the two projections of **a** and **b** (individually) into the direction of **c**, and why their sum must equal the projection of $\mathbf{a} + \mathbf{b}$ into the same **c**-direction. □

An alternate proof (of the original equivalence) would put the butts of $\mathbf{a}$ and $\mathbf{b}$ into the origin (they are free vectors, i.e. free to 'slide'), and out of all points along $\mathbf{b}$ [i.e. $t(b_1, b_2, b_3)$, where t is arbitrary], find the one which is closest to the tip of $\mathbf{a}$ (resulting in the $\mathbf{a} \to \mathbf{b}$ projection). This leads to minimizing the corresponding distance, namely $\sqrt{(a_1 - tb_1)^2 + (a_2 - tb_2)^2 + (a_3 - tb_3)^2}$ .The smallest value is achieved with $t_m = \frac{a_1b_1+a_2b_2+a_3b_3}{b_1^2+b_2^2+b_3^2}$ [by the usual procedure]. Thus, the *length* of this projection is $t_m|\mathbf{b}| = \frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{b_1^2+b_2^2+b_3^2}} = \frac{a_1b_1+a_2b_2+a_3b_3}{|\mathbf{b}|}$. This must equal to $|\mathbf{a}|\cos\gamma$, as we wanted to prove. □

▷ **Cross** (outer) [vector] **product**

(*notation*: $\mathbf{a} \times \mathbf{b}$) is defined as a *vector* whose length is $|\mathbf{a}|\cdot|\mathbf{b}|\cdot\sin\gamma$ (i.e. the area of a parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as two of its sides), whose direction is perpendicular (ORTHOGONAL) to each $\mathbf{a}$ and $\mathbf{b}$, and whose orientation is such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ follow the right-handed pattern [this makes the product *anti*-commutative, i.e. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$].

One way of visualizing its construction is this: project $\mathbf{a}$ into the plane perpendicular to $\mathbf{b}$ ($\equiv$ the blackboard, $\mathbf{b}$ is pointing inboard), rotate this projection by $+90^o$ (counterclockwise) and multiply the resulting vector by $|\mathbf{b}|$.

Also note that this product is *not* associative: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

The cross product is usually *computed* based on the following symbolic scheme:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3 \equiv$$

$$(a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$

[e.g. $(1, 3, -2) \times (4, -2, 1) = (-1, -9, -14)$].

The proof that the two definitions are identical rests on the validity of the distributive law: $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} \equiv \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$, which can be understood by visualizing $\mathbf{a}$ and $\mathbf{b}$ projected into a plane perpendicular to $\mathbf{c}$, constructing the vectors on each side of the equation and showing that they are identical. □

Optional: Another way of expressing the k^{th} component of $(\mathbf{a} \times \mathbf{b})$ is:

$$(\mathbf{a} \times \mathbf{b})_k = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \epsilon_{ijk}$$

(for $k = 1, 2, 3$), where ϵ_{ijk} [called a *fully antisymmetric* TENSOR] changes sign when any two indices are interchanged ($\Rightarrow \epsilon = 0$ unless i, j, k distinct) and $\epsilon_{123} = 1$ (this defines the rest).

One can show that

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{j\ell}\delta_{im}$$

(where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$; this is KRONECKER'S DELTA).

Based on this result, one can prove several useful formulas such as, for example:

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\bullet\mathbf{c})\mathbf{b} - (\mathbf{b}\bullet\mathbf{c})\mathbf{a}$$

Proof: The m^{th} component of the left hand side is $\sum\limits_{i,j,k,\ell} \epsilon_{ijk}a_ib_j\epsilon_{k\ell m}c_\ell = \sum\limits_{i,j,\ell}(\delta_{i\ell}\delta_{jm} - \delta_{j\ell}\delta_{im})a_ib_jc_\ell = \sum\limits_{\ell}(a_\ell b_m c_\ell - a_m b_\ell c_\ell)$ [the m^{th} component of the right hand side]. □

and

$$(\mathbf{a}\times\mathbf{b})\bullet(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\bullet\mathbf{c})(\mathbf{b}\bullet\mathbf{d}) - (\mathbf{a}\bullet\mathbf{d})(\mathbf{b}\bullet\mathbf{c})$$

having a similar proof. ⊗

▷ Triple product

of **a**, **b** and **c** is, by definition, equal to $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})$.

Computationally, this is identical to following determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, and it represents the volume of the parallelepiped with **a**, **b** and **c** being three of its sides (further multiplied by -1 if the three vectors constitute a *left-handed* set).

This implies that $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\bullet(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\bullet(\mathbf{a}\times\mathbf{b}) = -\mathbf{b}\bullet(\mathbf{a}\times\mathbf{c}) = -\mathbf{c}\bullet(\mathbf{b}\times\mathbf{a}) = \mathbf{a}\bullet(\mathbf{c}\times\mathbf{b})$ [its value does not change under *cyclic* permutation of the three vectors].

A useful application of the triple product is the following test: **a, b** and **c** are in the same plane (CO-PLANAR) iff $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) = 0$.

Another one is to compute the volume of an (arbitrary) tetrahedron. Note that if you use the three vectors as sides of the tetrahedron (instead of parallelepiped), its base will be half of the parallelepiped's, and its volume will thus be $\frac{\mathbf{a}\bullet(\mathbf{b}\times\mathbf{c})}{6}$. ■

In **general**: if you slide a (planar) base along a straight line to create a 3-D volume, this volume can be computed as the area of the base times the *perpendicular* height; if instead you create a 'cone' by running a straight line from each point of the base's boundary to the tip of the object, the corresponding volume will be 3 times smaller.

Proof of the last assertion: Volume is computed by $\int\limits_0^h A(x)\,dx$ where $A(x)$ is the area of the *cross-section* at height x, and h is the *total height* of a 3-D object. In our case $A(x) = A_0\cdot\frac{(x-h)^2}{h^2}$ where A_0 is the base area [right?]. This implies that the total volume equals $\frac{A_0}{h^2}\int\limits_0^h (x-h)^2\,dx = \frac{A_0}{h^2}\left[\frac{(x-h)^3}{3}\right]_{x=0}^h = \frac{A_0h}{3}$. □

Optional: ▶Rotation◀

of a coordinate system [to match it to someone else's, who uses the same origin but places the axes differently]. Suppose that *his* coordinates of *our* point (x, y, z) are (x', y', z') – what is the relationship between the two?

What is needed is some mathematical description of the corresponding ROTATION (to move our coordinates to his).At first one may (incorrectly) assume that a rotation is best represented by a vector [its direction being the axis of rotation, its length being the rotation angle]. The problem with such a description is this: one of the main operations we want to correctly describe is performing COMPOSITION (applying one after the other) of two or more rotations, and we need the corresponding mathematical 'machinery'. If we use the proposed vector representation of a rotation, the only 'composition' of two vectors we learned about is taking their cross product, which does *not* correspond to the composition of two rotations [which, unlike the cross product, *is* associative].

To find the proper way of representing rotations, we first realize that a rotation is a transformation (mapping) of points, symbolically: $\mathbf{r}' = \mathcal{R}(\mathbf{r})$ [as in Physics, we now use $\mathbf{r} \equiv (x, y, z)$ as a general notation for a point], This transformation is obviously *linear* [meaning $\mathcal{R}(c\mathbf{r}) = c\mathcal{R}(\mathbf{r})$ and $\mathcal{R}(\mathbf{r}_1 + \mathbf{r}_2) = \mathcal{R}(\mathbf{r}_1) + \mathcal{R}(\mathbf{r}_2)$, where c is a scalar]. We already know (from Linear Algebra) that a linear transformation of $\mathbf{r}$ corresponds to multiplying $\mathbf{r}$ (in its column form) by a 3×3 matrices [say $\mathbb{R}$], thus: $\mathbf{r}' = \mathbb{R}\mathbf{r}$.

But a rotation is a *special* case of a linear transformation; it preserves both lengths and angles (between vectors), which implies that it also preserves our *dot* product, i.e. $(\mathbf{r}_1')^T\mathbf{r}_2' \equiv \mathbf{r}_1^T\mathbf{r}_2$ (a matrix representation of the dot product) for any $\mathbf{r}_1$ and $\mathbf{r}_2$. This is the same as $\mathbf{r}_1^T\,\mathbb{R}^T\mathbb{R}\,\mathbf{r}_2 \equiv \mathbf{r}_1^T\mathbf{r}_2 \Rightarrow \mathbb{R}^T\mathbb{R} \equiv \mathbb{I}$ (such matrices are called ORTHOGONAL).

All this implies that rotations must be represented by orthogonal matrices. Now in reverse: Does each orthogonal matrix represent a rotation? The answer is 'no', orthogonal matrices allow the possibility of a REFLECTION (with respect to a plane), since it *also* preserves lengths and angles. To eliminate reflections (and be left with 'pure' rotations only), we have to further insist that $\det(\mathbb{R}) = +1$ (and not -1).

The matrix representation enables us to 'compose' two rotations by a simple matrix multiplication of the corresponding $\mathbb{R}_1$ and $\mathbb{R}_2$ (in reverse order), thus: $\mathbf{r}' = \mathbb{R}_2\mathbb{R}_1\mathbf{r}$. This operation is associative (even though non-commutative), in full agreement with what we already know about rotations.

Finding the orthogonal matrix which corresponds to a specific rotation is a fairly complicated procedure.

There is a recent mathematical formalism which simplifies all this (a rotation *is* represented by a vector), but it requires a rudimentary knowledge of QUOTERNION algebra (that is why it has not become widely used yet). $\otimes$

▶Straight Lines and Planes◀

(later to be extended to curved lines and surfaces).

There are two ways of defining a STRAIGHT LINE:

(i) **parametric representation**, i.e. $\mathbf{a} + \mathbf{b}\cdot t$ where $\mathbf{a}$ is an arbitrary point on the straight line and $\mathbf{b}$ is a vector along its direction, and t (the actual PARAMETER) is a scalar allowed to vary from $-\infty$ to $+\infty$

(ii) by **two linear equations**, e.g. $\begin{cases} 2x+3y-4z=6 \\ x-2y+z=-2 \end{cases}$ (effectively an intersection of two planes). ■

Neither description is unique (a headache when marking assignments).

Similarly, there are two ways of defining a PLANE:

(i) **parametric**, i.e. $\mathbf{a}+\mathbf{b}\cdot u+\mathbf{c}\cdot v$ where $\mathbf{a}$ is an arbitrary point in the plane, $\mathbf{b}$ and $\mathbf{c}$ are two nonparallel vectors within the plane, and u and v are scalar parameters varying over all possible real values

(ii) by a **single linear equation**, e.g. $2x+3y-4z=6$ [note that $(2,3,-4)$ is a vector *perpendicular* to the plane, its so called NORMAL – to prove it substitute two distinct points into the equation and subtract, getting the dot product of the connecting vector and $(2,3,-4)$, always equal to zero]. ■

Again, neither description is unique.

EXAMPLES:

1. Convert $\begin{cases} 3x+7y-4z=5 \\ 2x-3y+z=-4 \end{cases}$ to its parametric representation.

   Solution: The cross product of the two normals must point along the straight line, giving us $\mathbf{b}=(3,7,-4)\times(2,-3,1)=(-5,-11,-23)$. Solving the two equations with an arbitrary value of z (say $=0$) yields $\mathbf{a}=(-\frac{13}{23},\frac{22}{23},0)$.

   Answer: $(-\frac{13}{23}-5t,\frac{22}{23}-11t,-23t)$.

2. Find a equation of an (infinite) cylindrical surface with $(3-2t,1+3t,-4t)$ as its axis, and with the radius of 5.

   Solution: Let us first find an expression of the (shortest) distance from a point $\mathbf{r}\equiv(x,y,z)$ to a straight line $\mathbf{a}+\mathbf{b}\cdot t$ [bypassing minimization]. Visualize the vector $\mathbf{r}-\mathbf{a}$. We know that $|\mathbf{r}-\mathbf{a}|$ is its length, and that $(\mathbf{r}-\mathbf{a})\bullet\frac{\mathbf{b}}{|\mathbf{b}|}$ is the length of its projection into the straight line. By Pythagoras, the *direct* distance is $\sqrt{|\mathbf{r}-\mathbf{a}|^2-\left[(\mathbf{r}-\mathbf{a})\bullet\frac{\mathbf{b}}{|\mathbf{b}|}\right]^2}=\sqrt{(x-3)^2+(y-1)^2+z^2-\frac{(-2x+3y-4z+3)^2}{29}}$ (in our case). Making this equal to 5 yields the desired equation (square it to simplify).

   Answer: $(x-3)^2+(y-1)^2+z^2-\frac{(-2x+3y-4z+3)^2}{29}=25$.

3. What is the (shortest) distance from $\mathbf{r}=(6,2,-4)$ to $3x-4y+z=7$ [bypass minimization].

   Solution: $\mathbf{n}\bullet(\mathbf{r}-\mathbf{a})$, where $\mathbf{n}$ is the *unit* normal and $\mathbf{a}$ is an arbitrary point of the plane [found, in this case, by setting $x=y=0\Rightarrow(0,0,7)$].

   Answer: $\frac{(3,-4,1)}{\sqrt{9+16+1}}\bullet(6,2,-11)=-\frac{1}{\sqrt{26}}$ [the minus sign establishes on which side of the plane we are].

4. Find the (shortest) distance between $\mathbf{a}_1 + \mathbf{b}_1 \cdot t$ and $\mathbf{a}_2 + \mathbf{b}_2 \cdot t$ [bypassing minimization, as always].

   Solution: To find it, we have to move perpendicularly to both straight lines, i.e. along $\mathbf{b}_1 \times \mathbf{b}_2$. We also know that $\mathbf{a}_2 - \mathbf{a}_1$ is an arbitrary connection between the two lines. The projection of this vector into the direction of $\mathbf{b}_1 \times \mathbf{b}_2$ supplies (up to the sign) the answer: $(\mathbf{a}_2 - \mathbf{a}_1) \bullet \frac{\mathbf{b}_1 \times \mathbf{b}_2}{|\mathbf{b}_1 \times \mathbf{b}_2|}$ [visualize the situation by projecting the two straight lines into the blackboard so that they *look* parallel – always possible]. ■

▶Curves◀

are defined via their **parametric representation** $\mathbf{r}(t) \equiv [x(t), y(t), z(t)]$, where $x(t)$, $y(t)$ and $z(t)$ are arbitrary (continuous) functions of t (the parameter, ranging over some interval of real numbers).

EXAMPLE: $\mathbf{r}(t) = [\cos(t), \sin(t), t]$ is a HELIX centered on the z-axis, whose radius (when projected into the x-y plane) equals 1, with one full loop per 2π of vertical distance. The same $\mathbf{r}(t)$ can be also seen as a *motion* of a point-like particle, where t represents time. Note that $[\cos(2t), \sin(2t), 2t]$ represents a different motion (the particle is moving twice as fast), but the *same* curve (i.e. parametrization of a curve is far from unique). ■

▷ **Arc's length**

('arc' meaning a specific segment of the curve). The three-component (*vector*) distance travelled between time t and $t + dt$ (dt infinitesimal) is $\mathbf{r}(t+dt) - \mathbf{r}(t) \approx \mathbf{r}(t) + \dot{\mathbf{r}}(t)\, dt + ... - \mathbf{r}(t) = \dot{\mathbf{r}}(t)\, dt + ...$, where the dots stand for terms proportional to dt^2 and higher [these give zero contribution in the $dt \to 0$ limit], and $\dot{\mathbf{r}}(t)$ represents the *componentwise* differentiation with respect to t (the particle's VELOCITY). This converts to $|\dot{\mathbf{r}}(t)|\, dt + ...$ in terms of the actual *scalar* distance (length). Adding all these infinitesimal distances (from time a to time b – these should correspond to the arc's end points) results in

$$\int_a^b |\dot{\mathbf{r}}(t)|\, dt$$

which is the desired formula for the total length.

EXAMPLES:

1. Consider the helix of the previous example. The length of one of its complete loops (say from $t = 0$ to $t = 2\pi$) is thus $\int_0^{2\pi} |[-\sin(t), \cos(t), 1]|\, dt = \int_0^{2\pi} \sqrt{\sin(t)^2 + \cos(t)^2 + 1}\, dt = 2\pi\sqrt{2}$.

2. The intersect of $x^2 + y^2 = 9$ (a cylinder) and $3x - 4y + 7z = 2$ (a plane) is an ellipse. How long is it?

   Solution: First we need to parametrize it, thus: $\mathbf{r}(t) = [3\cos(t), 3\sin(t), \frac{2-9\cos(t)+12\sin(t)}{7}]$ where $t \in [0, 2\pi)$.

   Answer: $\int_0^{2\pi} |\dot{\mathbf{r}}|\, dt = \int_0^{2\pi} \sqrt{9 + \left(\frac{9\sin t + 12\cos t}{7}\right)^2}\, dt$ which is an integration we cannot carry out analytically (just to remind ourselves that this can frequently happen). Numerically (using Maple), this equals 21.062.

▷ A **tangent** (straight) **line**

to a curve, at a point $\mathbf{r}(t_0)$ [t_0 being a specific value of the parameter] passes through $\mathbf{r}(t_0)$, and has the direction of $\dot{\mathbf{r}}(t_0)$ [the velocity]. Its parametric representation will be thus

$$\mathbf{r}(t_0) + \dot{\mathbf{r}}(t_0) \cdot u$$

[where u is the parameter now, just to differentiate].

EXAMPLE: Using the same helix, at $t = 0$ its tangent line is $[1, u, u]$. ■

▷ When $\mathbf{r}(t)$ is seen as a motion of a particle, $\dot{\mathbf{r}}(t) \equiv \mathbf{v}(t)$ gives the particle's (instantaneous, 3-D) **velocity**. $|\dot{\mathbf{r}}(t)|$ then yields its (*scalar*) **speed** [the speedometer reading]. It is convenient to rewrite $\mathbf{v}(t)$ as $|\dot{\mathbf{r}}(t)| \cdot \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} \equiv$

$$|\dot{\mathbf{r}}(t)| \cdot \mathbf{u}(t)$$

[a product of its speed and *unit* direction].

The corresponding (3-D) **acceleration**
is simply $\mathbf{a}(t) \equiv \ddot{\mathbf{r}}(t)$ [a double t-derivative]. It is more meaningful to decompose it into its 'TANGENTIAL' [the one observed on the speedometer, pushing you back into your seat] and 'NORMAL' [observed even at constant speeds, pushing you sideways – perpendicular to the motion] components. This is achieved by the product rule: $\frac{d\mathbf{v}(t)}{dt} = \frac{d|\dot{\mathbf{r}}(t)|}{dt} \cdot \mathbf{u}(t) + |\dot{\mathbf{r}}(t)| \cdot \frac{d\mathbf{u}(t)}{dt}$ [tangential and normal, respectively]. $\frac{d|\dot{\mathbf{r}}(t)|}{dt}$ can be simplified to $\frac{d}{dt}\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} = \frac{1}{2} \cdot \frac{2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{z}\ddot{z}}{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}} = \frac{\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}}}{|\dot{\mathbf{r}}(t)|} =$

$$\mathbf{u} \bullet \ddot{\mathbf{r}} \qquad \text{(tangential magnitude)}$$

The normal acceleration is then most easily computed from

$$\ddot{\mathbf{r}} - (\mathbf{u} \bullet \ddot{\mathbf{r}})\mathbf{u} \qquad \text{(normal)}$$

[full minus tangential]. In this form it is trivial to verify that the normal acceleration is perpendicular to $\mathbf{u}$.

EXAMPLE: For our helix at $t = 0$, the speed is $\sqrt{2}$, $\mathbf{u} = [0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ and $\ddot{\mathbf{r}} = [-1, 0, 0] \Rightarrow$ zero tangential acceleration and $[-1, 0, 0]$ normal acceleration. ■

▷ When interested in the *geometric* properties of a curve only, it is convenient to make its parametrization *unique* by introducing a **special parameter** s (instead of t) which measures the actual *length travelled* along the curve, i.e.

$$s(t) = \int_0^t |\dot{\mathbf{r}}(t)| \; dt$$

where $\mathbf{r}(t)$ is the old parametrization.

Unfortunately, to carry out the details of such a 'reparametrization' is normally too difficult [to eliminate t, we would have to solve the previous equation for t – but we don't know how to solve general equations]. Yet, the idea of this new 'uniform' (in the sense of the corresponding motion) parameter s is still quite helpful, when we realize that the previous equation is equivalent to

$$\frac{ds}{dt} = |\dot{\mathbf{r}}(t)|$$

This further implies that, even though we don't have an explicit formula for $s(t)$, we know how to *differentiate* with respect to s, as

$$\frac{d}{ds} \equiv \frac{\frac{d}{dt}}{\frac{ds}{dt}} \equiv \frac{\frac{d}{dt}}{|\dot{\mathbf{r}}(t)|}$$

Note that our old $\mathbf{u} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|}$ [the unit velocity direction] can thus be defined simply as $\frac{d\mathbf{r}}{ds} \equiv \mathbf{r}'$ [prime will imply s-differentiation].

Using this new parameter s, we now *define* a few interesting geometrical properties (describing a curve and its behavior in space); we will immediately 'translate' these into the t-'language', as we normally parametrize curves by t and not s:

▷ Curvature

Let us first compute $\frac{d\mathbf{u}}{ds} \equiv \mathbf{r}''$ which corresponds to the rate of change of the unit direction per (scalar) distance travelled. The result is a vector which is always perpendicular to $\mathbf{u}$, as we will show shortly.

Curvature κ is the *magnitude* of this $\mathbf{r}''$, and corresponds, geometrically, to the reciprocal of the radius of a TANGENT CIRCLE to the curve at a point [a circle with the same $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ – 6 independent conditions].

The main thing now is to figure out is: how do we compute curvature when our curve has the usual t-parametrization? This is not too difficult, as $\dfrac{d\mathbf{u}(s)}{ds} =$ $\dfrac{\frac{d\mathbf{u}(t)}{dt}}{|\dot{\mathbf{r}}(t)|} = \dfrac{\frac{d}{dt}\frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}}{|\dot{\mathbf{r}}|} = \dfrac{\ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^2} - \dfrac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|^3} \cdot (\mathbf{u} \bullet \ddot{\mathbf{r}})$ [since $\dfrac{d|\dot{\mathbf{r}}|}{dt} = \frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = (\mathbf{u} \bullet \ddot{\mathbf{r}})$] =

$$\frac{\ddot{\mathbf{r}}(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}}) - \dot{\mathbf{r}}(\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}})}{|\dot{\mathbf{r}}|^4}$$

[this is easily seen to be $\dot{\mathbf{r}}$ perpendicular, as claimed].

To get κ, we need the corresponding magnitude:

$$\sqrt{\frac{(\ddot{\mathbf{r}}\bullet\ddot{\mathbf{r}})(\dot{\mathbf{r}}\bullet\dot{\mathbf{r}})^2+(\dot{\mathbf{r}}\bullet\dot{\mathbf{r}})(\dot{\mathbf{r}}\bullet\ddot{\mathbf{r}})^2-2(\dot{\mathbf{r}}\bullet\ddot{\mathbf{r}})^2(\dot{\mathbf{r}}\bullet\dot{\mathbf{r}})}{(\dot{\mathbf{r}}\bullet\dot{\mathbf{r}})^4}} =$$

$$\sqrt{\frac{(\dot{\mathbf{r}}\bullet\dot{\mathbf{r}})(\ddot{\mathbf{r}}\bullet\ddot{\mathbf{r}})-(\dot{\mathbf{r}}\bullet\ddot{\mathbf{r}})^2}{(\dot{\mathbf{r}}\bullet\dot{\mathbf{r}})^3}}$$

This is the final formula for computing curvature.

EXAMPLE: For the same old helix, $(\dot{\mathbf{r}}\bullet\dot{\mathbf{r}})=2$, $(\ddot{\mathbf{r}}\bullet\ddot{\mathbf{r}})=1$, and $(\dot{\mathbf{r}}\bullet\ddot{\mathbf{r}})=0\Rightarrow$ $\kappa=\sqrt{\frac{1}{2^2}}=\frac{1}{2}$ [the same for all points of the helix – that seems to make sense; the tangent circles all have a radius of 2]. ■

▷ A few **related definitions**

From what we already know $\mathbf{r}''=\kappa\cdot\mathbf{p}$ where $\mathbf{p}$ is a *unit* vector we will call PRINCIPAL NORMAL, automatically orthogonal to $\mathbf{u}$ and pointing towards the tangent circle's center. Furthermore, $\mathbf{b}=\mathbf{u}\times\mathbf{p}$ must thus be yet another *unit* vector, orthogonal to both $\mathbf{u}$ and $\mathbf{p}$. It is called the BINORMAL vector (perpendicular to the tangent circle's plane).

One can show that the *rate of change* of $\mathbf{b}$ (per unit distance travelled), namely $\mathbf{b}'$ is a vector in the direction of $\mathbf{p}$, i.e. $\mathbf{b}'=-\tau\cdot\mathbf{p}$, where τ defines the so called **torsion** ('twist') of the curve at the corresponding point [τ is thus either + or − of the corresponding magnitude, the extra minus sign is just a convention].

Note that knowing a curve's curvature and torsion, we can 'reconstruct' the curve (by solving the corresponding set of differential equations), but we will not go into that.

We now derive a formula for computing τ based on the usual $\mathbf{r}(t)$-parametrization.

First: $\mathbf{b}'=\mathbf{u}'\times\mathbf{p}+\mathbf{u}\times\mathbf{p}'=\mathbf{0}+\mathbf{u}\times\left(\frac{\mathbf{u}'}{\kappa}\right)'$ [since $\mathbf{u}'\equiv\kappa\mathbf{p}$].

Then: $\tau=-\mathbf{p}\bullet\mathbf{b}'=-\left(\frac{\mathbf{u}'}{\kappa}\right)\bullet\left[\mathbf{u}\times\left(\frac{\mathbf{u}'}{\kappa}\right)'\right]=-\dfrac{\mathbf{u}'\bullet(\mathbf{u}\times\mathbf{u}'')}{\kappa^2}=\dfrac{\mathbf{u}\bullet(\mathbf{u}'\times\mathbf{u}'')}{\kappa^2}$.

And finally: $\mathbf{u}=\mathbf{r}'=\dot{\mathbf{r}}\frac{dt}{ds}$, $\mathbf{u}'=\mathbf{r}''=\ddot{\mathbf{r}}(\frac{dt}{ds})^2+\dot{\mathbf{r}}\frac{d^2t}{ds^2}$ and $\mathbf{u}''=\mathbf{r}'''=\dddot{\mathbf{r}}\left(\frac{dt}{ds}\right)^3+3\ddot{\mathbf{r}}\frac{dt}{ds}\cdot\frac{d^2t}{ds^2}+\dot{\mathbf{r}}\frac{d^3t}{ds^3}$.

Putting it together [and realizing that, whenever identical vectors 'meet' in a triple product, the result is zero], we get $\tau=\frac{\dot{\mathbf{r}}\bullet(\ddot{\mathbf{r}}\times\dddot{\mathbf{r}})}{\kappa^2}\left(\frac{dt}{ds}\right)^6=$ [since $\frac{dt}{ds}=\frac{1}{|\dot{\mathbf{r}}|}$]

$$\frac{\dot{\mathbf{r}}\bullet(\ddot{\mathbf{r}}\times\dddot{\mathbf{r}})}{(\dot{\mathbf{r}}\bullet\dot{\mathbf{r}})(\ddot{\mathbf{r}}\bullet\ddot{\mathbf{r}})-(\dot{\mathbf{r}}\bullet\ddot{\mathbf{r}})^2}$$

which is our final formula for computing torsion.

Both the original definition and the final formula clearly imply that a *planar* curve has a *zero* torsion (identically).

EXAMPLE: For the helix $\dddot{\mathbf{r}}=(\sin t,-\cos t,0)\Rightarrow\tau=\frac{1}{2}$. ■

In the next chapter we will introduce SURFACES (two-dimensional structures in 3-D; curves are of course one-dimensional) and explore the related issues. But now we interrupt this line of development to introduce

Fields

A **scalar field** is just a fancy name for a function of x, y and z [i.e. to each point in space we attach a single value, say its temperature], e.g. $f(x,y,z) = \frac{x(y+3)}{z}$.

A **vector field** assigns, to each point in space, a vector value (i.e. three numbers rather than just one). Mathematically, this corresponds to having three functions of x, y and z which are seen as three components of a vector, thus: $\mathbf{g}(x,y,z) \equiv [g_1(x,y,z), g_2(x,y,z), g_3(x,y,z)]$, e.g. $[xy, \frac{z-3}{x}, \frac{y(x-4)}{z^2}]$ Physically, this may represent a field of some force, permeating the space.

An **operator** is a 'prescription' which takes a field and modifies it (usually, by computing its derivatives, in which case it is called a *differential* operator) to return another field. To avoid further difficulties relating to differential operators, we have to assume that our fields are sufficiently 'SMOOTH' (i.e. not only continuous, but also differentiable at each point).

The most important cases of operators (acting in 3-D space) are:

▶Gradient◀

which converts a scalar field $f(x,y,z)$ into the following vector field

$$(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

$\equiv$ [**notation**] $\boldsymbol{\nabla} f(x,y,z)$. The $\boldsymbol{\nabla}$-*operator* is usually called 'DEL' (sometimes 'nabla'), and has three components, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$, i.e. it can be considered to have *vector* attributes.

It yields the direction of the fastest increase in $f(x,y,z)$ when starting at (x,y,z); its magnitude provides the corresponding rate (per unit length). This can be seen by rewriting the generalized Taylor expansion of f at $\mathbf{r}$, thus: $f(\mathbf{r}+\mathbf{h}) = f(\mathbf{r}) + \mathbf{h} \bullet \boldsymbol{\nabla} f(\mathbf{r}) +$ quadratic (in $\mathbf{h}$-components) and higher-order terms. When $\mathbf{h}$ is a *unit* vector, $\mathbf{h} \bullet \boldsymbol{\nabla} f(\mathbf{r})$ provides a so called **directional derivative** of f, i.e. the rate of its increase in the $\mathbf{h}$-direction [obviously the largest when $\mathbf{h}$ and $\boldsymbol{\nabla} f$ are parallel].

An interesting **geometrical application** is this: $f(x,y,z) = c$ [constant] usually defines a surface (a 3-D 'contour' of f – a simple extension of the $f(x,y) = c$ idea). The gradient, evaluated at a point of such a surface, is obviously *normal* (perpendicular) to the surface at that point.

EXAMPLE: Find the normal direction to $z^2 = 4(x^2+y^2)$ [a cone] at $(1,0,2)$ [this must lie on the given surface, check].

Solution: $f \equiv 4(x^2+y^2) - z^2 = 0$ defines the surface. $\boldsymbol{\nabla} f = (8x, 8y, -2z)$, evaluated at $(1,0,2)$ yields $(8,0,-4)$, which is the answer. One may like to convert it to a *unit* vector, and spell out its orientation (either inward or outward). ■

Application to Physics: If $\mathbf{r}(t)$ represents a *motion* of a particle and $f(x,y,z)$ a temperature of the 3-D media in which the particle moves, $\dot{\mathbf{r}} \bullet \boldsymbol{\nabla} f[\mathbf{r}(t)]$ is the RATE OF CHANGE (per unit of *time*) of temperature as the particle experiences it [nothing but a chain rule]. To convert this into a spacial (per unit *length*) rate, one would have to divide the previous expression by $|\dot{\mathbf{r}}|$.

►Divergence◄

converts a vector field $\mathbf{g}(\mathbf{r})$ to the following scalar field:

$$\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}$$

$\equiv$ [**symbolically**] $\boldsymbol{\nabla} \bullet \mathbf{g}(\mathbf{r})$.

Its significance (to Physics) lies in the following **interpretation**: If $\mathbf{g}$ represents some FLOW [the direction and rate of a motion of some continuous substance in space; the rate being established by measuring mass/sec./cm.^{2} through an infinitesimal area perpendicular to its direction], then the divergence tells us the rate of *mass loss* from an (infinitesimal) volume at each point, *per volume* [mass/sec./cm.^{3}]. This can be seen by surrounding the point by an (infinitesimal) cube, and figuring out the in/out flow through each of its sides [$h^2 g_1(x + \frac{h}{2}, y, z)$ is the outflow from one of them, etc.].

Optional: A flow (also called FLUX) is usually expressed as a product of the substance' density $\rho(\mathbf{r})$ [measured as mass/cm.^{3}, obviously a scalar field] and its velocity $\mathbf{v}(\mathbf{r})$ [measured in cm./sec., obviously a vector field]. The equation $\boldsymbol{\nabla}\bullet[\rho\mathbf{v}] + \dfrac{\partial \rho}{\partial t} = 0$ then expresses the conservation-of-mass law – no mass is being lost or created (no SINKS nor SOURCES), any outflow of mass results in a corresponding reduction of density. Here we have assumed that our ρ and $\mathbf{v}$ fields are functions of not only x, y and z, but also of t (time), as often done in Physics. ⊗

EXAMPLE: Find $\boldsymbol{\nabla}\bullet(x^2, y^2, z^2)$. Answer: $2x + 2y + 2z$. ■

►Curl◄

(sometimes also called **rotation**), applied to a vector field $\mathbf{g}$, converts it to yet another vector field symbolically defined by $\boldsymbol{\nabla} \times \mathbf{g}$, i.e.

$$[\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}, \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}, \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}]$$

If $\mathbf{g}$ represents a flow, $Curl(\mathbf{g})$ can then be visualized by holding an imaginary paddle-wheel at each point to see how fast the wheel rotates (its axis at the fastest rotation yields the curl's direction, the torque establishes the corresponding magnitude).

EXAMPLE: $Curl(x, yz, -x^2 - z^2) = (-y, 2x, 0)$. ■

▷ One can easily prove the following **trivial identities**:

$$\begin{aligned} Curl\,[Grad\,(f)] &\equiv \mathbf{0} \\ Div\,[Curl\,(\mathbf{g})] &= 0 \end{aligned}$$

There are also several nontrivial identities, for illustration we mention one only:

$$Div(\mathbf{g}_1 \times \mathbf{g}_2) = \mathbf{g}_2 \bullet Curl(\mathbf{g}_1) - \mathbf{g}_1 \bullet Curl(\mathbf{g}_2)$$

Optional: Divergence and gradient are frequently applied, consecutively, to a scalar field f, to create a new scalar field $Div[Grad(f)] \equiv \triangle f =$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

(where $\triangle$ is the so called **Laplace operator**). It measures how much the value of f (at each point) deviates from its average over some infinitesimal surface [visualize a cube] centered at the point, per the surface's area (the exact answer is obtained in the limit, as the size of the cube approaches zero). $\otimes$

Optional: Curvilinear coordinates

(such as, for example, the SPHERICAL COORDINATES r, θ and φ) is a set of three new *independent* variables [replacing the old (x, y, z)], each expressed as a function of x, y and z, and used to locate points in (3-D) space [by inverting the transformation].

Varying only the first of the *new* 'coordinates' (keeping the other two fixed) results in a corresponding **coordinate curve** (the changing coordinate becomes its parameter – the old t) whose *unit* direction is labelled $\mathbf{e}_r$ (similarly for the other two), and whose 'speed' (i.e. distance travelled per unit change of the new coordinate) is called h_r [both $\mathbf{e}_r$ and h_r are functions of location; they can be easily established geometrically].

When $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ remain perpendicular to each other at every point, the new coordinate system is called **orthogonal** (the case of spherical coordinates). All our subsequent results apply to orthogonal coordinates only.

Fields can be easily transformed to new coordinates, all it takes is to express x, y and z in terms of r, θ and φ. How do we compute $Grad$, Div and $Curl$ in the new coordinates, so that they agree with the old, rectangular-coordinate results?

First of all, $Grad$ and $Curl$ will be expressed in terms of the new, curvilinear axes $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$, instead of the original $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Each component of the new, curvilinear $Grad(f)$ should express the (instantaneous) rate of increase of f, when moving along the respective $\mathbf{e}$, per distance travelled (let us call this distance d). Thus, when we increase the value of r and start moving along $\mathbf{e}_r$, we obtain: $\dfrac{\partial f}{\partial d} = \dfrac{\partial f}{\partial r} \cdot \dfrac{\partial r}{\partial d} \equiv \dfrac{\frac{\partial f}{\partial r}}{h_r}$, by our previous definition of the h-functions. [Similarly for θ and φ.] The full gradient is thus

$$Grad(f) = \frac{\mathbf{e}_r}{h_r} \cdot \frac{\partial f}{\partial r} + \frac{\mathbf{e}_\theta}{h_\theta} \cdot \frac{\partial f}{\partial \theta} + \frac{\mathbf{e}_\varphi}{h_\varphi} \cdot \frac{\partial f}{\partial \varphi}$$

[note that, for spherical coordinates, $h_r = 1$, $h_\theta = r$ and $h_\varphi = r \sin\theta$].

To get $Div(\mathbf{g})$, we note that an infinitesimal volume ('near-cube') built by increasing r to $r + dr$, θ to $\theta + d\theta$ and φ to $\varphi + d\varphi$ has sides of length $h_r dr$, $h_\theta d\theta$ and $h_\varphi d\varphi$, faces of area $h_r h_\theta dr d\varphi$, $h_r h_\theta dr d\theta$ and $h_\theta h_\varphi d\theta d\varphi$, and volume of size $h_r h_\theta h_\varphi dr d\theta d\varphi$. One can easily see that $h_\theta h_\varphi g_r d\theta d\varphi$ is the total flow (FLUX) through one of the sides; $\frac{\partial}{\partial r}\left(h_\theta h_\varphi g_r\right) d\theta d\varphi dr$ is then the corresponding flux difference between the two opposite sides. Adding the three contributions and dividing by the total volume yields:

$$Div(\mathbf{g}) = \frac{1}{h_r h_\theta h_\varphi}\left[\frac{\partial}{\partial r}\left(h_\theta h_\varphi g_r\right) + \frac{\partial}{\partial \theta}\left(h_r h_\varphi g_\theta\right) + \frac{\partial}{\partial \varphi}\left(h_r h_\theta g_\varphi\right)\right]$$

In the case of spherical coordinates this reduces to:
$\frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta g_r\right) + \frac{\partial}{\partial \theta}\left(r \sin\theta g_\theta\right) + \frac{\partial}{\partial \varphi}\left(r g_\varphi\right)\right]$, yielding, for the corresponding **Laplacian**

$$Div[Grad(f)] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

Understanding this is essential when computing (quantum mechanically) the energy levels (eigenvalues) of Hydrogen atom.

Similarly one can derive a formula for $Curl(\mathbf{g})$, which we do not quote here (see your textbook). $\otimes$

Chapter 8 FUNCTIONS IN 3-D – INTEGRATION

Line Integrals

are of two types:

▶Scalar (Type I) Integrals◀

where we are given a (scalar) function $f(x, y, z)$ and a curve $\mathbf{r}(t)$, and need to integrate f over an arc of the curve (which now assumes the rôle of the x-axis). All it takes is to add the areas of the individual 'rectangles' of base $|\dot{\mathbf{r}}|\,dt$ and 'height' [which, unfortunately, has to be pictured in an extra 4^{th} dimension] $f[\mathbf{r}(t)]$, ending up with

$$\int_a^b f[\mathbf{r}(t)] \cdot |\dot{\mathbf{r}}(t)|\,dt \tag{LI}$$

which is just an ordinary (scalar) integral of a single variable t. Note that the result must be *independent* of the actual curve parametrization.

In this context we should mention that all our curves are PIECE-WISE SMOOTH, i.e. continuous, and consisting of one or more differentiable pieces (e.g. a square).

This kind of integration can used for *(spacial)* **averaging** of the f-values (over a segment of a curve). All we have to do is to divide the above integral by the arc's length $\int_a^b |\dot{\mathbf{r}}(t)|\,dt$:

$$\overline{f}_{sp} = \frac{\int_a^b f[\mathbf{r}(t)] \cdot |\dot{\mathbf{r}}(t)|\,dt}{\int_a^b |\dot{\mathbf{r}}(t)|\,dt}$$

To average in *time* (taking $\mathbf{r}(t)$ to be a motion of a particle) one would do

$$\overline{f}_{tm} = \frac{\int_a^b f[\mathbf{r}(t)]\,dt}{b-a}$$

instead.

The **symbolic notation** for this integral is

$$\int_C f(\mathbf{r})\,ds$$

s being the special unique parameter which corresponds to the 'distance travelled', and C stands for a specific segment of a curve. To evaluate this integral, we normally use a convenient (arbitrary) parametrization of the curve (the result must be the same), and carry out the integration in terms of t, using (LI).

Two other possible **applications** are:

1. **Center of mass** of a wire-like object of uniform mass density:

$$\left[\frac{\int_C x\,ds}{\int_C ds}, \frac{\int_C y\,ds}{\int_C ds}, \frac{\int_C z\,ds}{\int_C ds}\right]$$

   [the denominator is the total length L].

2. **Moment of inertia** of any such an object:

$$\frac{M}{L}\int_C d^2 \cdot ds$$

   where $d(x,y,z)$ is distance from the axis of rotation. [Angular acceleration is torque divided by moment of inertia]. ■

EXAMPLES:

- Evaluate $\int_C (x^2+y^2+z^2)^2\,ds$ where $C \equiv (\cos t, \sin t, 3t)$ with $t \in (0, 2\pi)$ [one loop of a helix].

  Solution: $\int_0^{2\pi} (\cos^2 t + \sin^2 t + 9t^2)^2 \sqrt{(-\sin t)^2 + (\cos t)^2 + 9}\,dt = \sqrt{10}\int_0^{2\pi}(1 + 18t^2 + 81t^4)\,dt = \sqrt{10}\left[t + 6t^3 + \frac{81}{5}t^5\right]_0^{2\pi} = 5.0639 \times 10^5$.

- Find the center of mass of a half circle (the circumference only) of radius a.

  Solution: $\mathbf{r}(t) = [a\cos t, a\sin t, 0] \Rightarrow \int_C y\,ds = a^2\int_0^{\pi} \sin t\,dt = 2a^2$.

  Answer: The center of mass is at $[0, \frac{2a^2}{\pi a}, 0] = [0, 0.63662a, 0]$.

- Find the moment of inertia of a circle (circumference) of mass M and radius a with respect to an axis passing through its center and two of its points.

  Solution: Using $\mathbf{r}(t) = [a\cos t, a\sin t, 0]$ and y as the axis, we get $\int_C (x^2 + z^2)\,ds = a^3\int_0^{2\pi}\cos^2 t\,dt = \pi a^3$.

  Answer: $\frac{M}{2\pi a} \cdot \pi a^3 = \frac{Ma^2}{2}$.

►Vector (Type II) Integrals◄

Here, we are given a vector function $\mathbf{g}(x,y,z)$ [i.e. effectively three functions g_1, g_2 and g_3] which represents a *force* on a point particle at (x,y,z), and a curve $\mathbf{r}(t)$ which represents the particle's 'motion'. We know (from Physics) that, when the particle is moved by an infinitesimal amount $d\mathbf{r}$, the energy it extracts from the field equals $\mathbf{g} \bullet d\mathbf{r}$ [when negative, the magnitude is the amount of work needed to make it move]. This is independent of the actual speed at which the move is made.

The **total energy** thus extracted (or, with a minus sign, the work needed) when a particle moves over a segment C is, *symbolically*,

$$\int_C \mathbf{g}(\mathbf{r}) \bullet d\mathbf{r}$$

[$\int_C g_1 dx + g_2 dy + g_3 dz$ is an alternate notation] and can be computed by parametrizing the curve (any way we like – the result is independent of the parametrization, i.e. the actual motion of the particle) and finding

$$\int_a^b \mathbf{g}[\mathbf{r}(t)] \bullet \dot{\mathbf{r}}(t)\, dt \tag{LII}$$

EXAMPLE: Evaluate $\int_C (5z, xy, x^2 z) \bullet d\mathbf{r}$ where $C \equiv (t, t, t^2)$, $t \in (0, 1)$.

Solution: $\int_0^1 (5t^2, t^2, t^4) \bullet (1, 1, 2t)\, dt = \int_0^1 (6t^2 + 2t^5)\, dt = \frac{7}{3} = 2.3333.$ ■

Note that, in general, the integral is PATH DEPENDENT, i.e. connecting the same two points by a different curve results in two different answers.

EXAMPLE: Compute the same $\int_C (5z, xy, x^2 z) \bullet d\mathbf{r}$, where now $C \equiv (t, t, t)$, $t \in (0, 1)$.

Solution: $\int_0^1 (5t, t^2, t^3) \bullet (1, 1, 1)\, dt = \int_0^1 (5t + t^2 + t^3)\, dt = \frac{37}{12} = 3.0833.$ ■

Could there be a *special type* of vector fields to make all such vector integrals

▷**Path Independent**

The answer is yes, this happens for any $\mathbf{g}$ which can be written as

$$\boldsymbol{\nabla} f(x, y, z)$$

[a gradient of a scalar field, which is called the corresponding POTENTIAL; $\mathbf{g}$ is then called a CONSERVATIVE vector field].

Proof: $\int_C (\boldsymbol{\nabla} f) \bullet d\mathbf{r} = \int_a^b (\boldsymbol{\nabla} f[\mathbf{r}(t)]) \bullet \dot{\mathbf{r}}(t)\, dt = [\leftarrow \text{chain rule}] \int_a^b \frac{df[\mathbf{r}(t)]}{dt} dt = f[\mathbf{r}(b)] - f[\mathbf{r}(a)].$ □

But how can we establish whether a given $\mathbf{g}$ is conservative? Easily, the sufficient and necessary **condition** is

$$Curl(\mathbf{g}) \equiv \mathbf{0}$$

Proof: $\mathbf{g} = \boldsymbol{\nabla} f$ clearly implies that $Curl(\mathbf{g}) \equiv \mathbf{0}$.

Now the reverse: Given such a $\mathbf{g}$, we construct (as discussed in the subsequent example) $f = \int g_1 dx + \int g_2 dy - \int \left(\int \frac{\partial g_1}{\partial y} dx \right) dy + \int g_3 dz - \int \left(\int \frac{\partial g_1}{\partial z} dx \right) dz - \int \left(\int \frac{\partial g_2}{\partial z} dy \right) dz + \int \left[\int \left(\int \frac{\partial^2 g_1}{\partial y \partial z} dx \right) dy \right] dz$.

This implies: $\frac{\partial f}{\partial x} = g_1 + \int \frac{\partial g_1}{\partial y} dy - \int \frac{\partial g_1}{\partial y} dy + \int \frac{\partial g_1}{\partial z} dz - \int \frac{\partial g_1}{\partial z} dz - \int \left(\int \frac{\partial^2 g_1}{\partial y \partial z} dy \right) dz + \int \left[\int \frac{\partial^2 g_1}{\partial y \partial z} dy \right] dz \equiv g_1$.

Similarly, we can show $\frac{\partial f}{\partial y} = g_2$ and $\frac{\partial f}{\partial z} = g_3$. □

Note that when $\mathbf{g}$ is conservative, all we need to specify is the starting and final point of the arc (how you connect them is irrelevant, as long as you avoid an occasional singularity). We can then use the following **notation**:

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{g}(\mathbf{r}) \bullet d\mathbf{r}$$

which gives you a strong hint that $\mathbf{g}$ is conservative (the notation would not make sense otherwise).

EXAMPLE: Evaluate $\int\limits_{(0,0,0)}^{(1,\frac{\pi}{4},2)} 2xyz^2 dx + [x^2 z^2 + z\cos(yz)]dy + [2x^2 yz + y\cos(yz)]dz$.

Solution: This is what we used to call 'exact differential form', extended to three independent variables. We solve it by integrating g_1 with respect to x [calling the result f_1], adding $g_2 - \frac{\partial f_1}{\partial y}$ integrated with respect to y [call the overall answer f_2], then adding the z integral of $g_3 - \frac{\partial f_2}{\partial z}$, to get the final f. In our case, this yields $x^2 y\, z^2$ for f_1, $x^2 yz^2 + \sin(yz)$ for $f_2 \equiv f$, as nothing is added in the last step. Thus $f(x, y, z) = x^2 yz^2 + \sin(yz)$ [check].

Answer: $f(1, \frac{\pi}{4}, 2) - f(0, 0, 0) = 1 + \pi = 4.1416$. ■

Optional: We mention in passing that, similarly, $Div(\mathbf{g}) = 0 \Leftrightarrow$ there is a vector field $\mathbf{h}$ say such that $\mathbf{g} \equiv Curl(\mathbf{h})$ [$\mathbf{g}$ is then called PURELY ROTATIONAL]. Any vector field $\mathbf{g}$ can be written as $Grad(f) + Curl(\mathbf{h})$, i.e. decomposed into its conservative and purely rotational part. ⊗

Double integrals

can be evaluated by two consecutive (univariate) integrations, the first with respect to x, over its *conditional* range given y, the second with respect to y, over its *marginal* range (or the other way round, the two answers must agree).

EXAMPLES:

- To integrate over the $\left\{ \begin{array}{l} x > 0 \\ y > 0 \\ x + y < 1 \end{array} \right.$ triangle, we first do $\int\limits_0^{1-y} ... \, dx$ followed by $\int\limits_0^1 ... \, dy$ (or $\int\limits_0^{1-x} ... \, dy$ followed by $\int\limits_0^1 ... \, dx$).

- To integrate over $0 < y < \frac{1}{x}$, where $1 < x < 3$, we can do either $\int_1^3 \int_0^{\frac{1}{x}} ... \, dy \, dx$ or $\int_{\frac{1}{3}}^1 \int_1^{\frac{1}{y}} ... \, dx \, dy + \int_0^{\frac{1}{3}} \int_1^3 ... \, dx \, dy$ [only a graph of the region can reveal why it is so].

- $\iint\limits_{x^2+y^2<1} y^2 \, dx \, dy = \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2 \, dx \right) dy = \int_{-1}^{1} 2y^2 \sqrt{1-y^2} dy =$
$\left[\frac{1}{4} \arcsin y + \frac{1}{4} y \sqrt{1-y^2} - 12y(1-y^2)^{\frac{3}{2}} \right]_{y=-1}^{1} = \frac{\pi}{4}$. ■

The last of these double integrals can be simplified by introducing

►Polar Coordinates◄

(effectively a change of variables, from the old x, y, to a new pair of r, φ) by:

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \end{aligned}$$

One has to remember that $dx \, dy$ of the double integration must be replaced by $dr \, d\varphi$, further *multiplied* by the **Jacobian** of the transformation, namely the absolute value of

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{dr} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix}$$

In our case (of polar coordinates) this equals to $\begin{vmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{vmatrix} = r.$

EXAMPLE: $\iint\limits_{x^2+y^2<1} y^2 \, dx \, dy = \int_0^{2\pi} \int_0^1 r^2 \sin^2 \varphi \cdot r \, dr \, d\varphi$ [note how, in polar coordinates, the region of integration is a simple 'rectangle' and the double integral becomes separable] $= \int_0^1 r^3 dr \times \int_0^{2\pi} \sin^2 \varphi \, d\varphi = \frac{1}{4} \times \left[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4} \right]_0^{2\pi} = \frac{\pi}{4}$ [a lot easier than the direct integration above]. ■

Similarly to polar coordinates, one can introduce any other set of **new variables** to simplify the integration (the actual form of the transformation would be normally suggested to you).

EXAMPLE: $\iint\limits_{\mathcal{R}} y^2 \, dx \, dy$ where $\mathcal{R}$ is a square with corners at $(0,1)$, $(1,0)$, $(0,-1)$ and $(-1,0)$.

Introducing u, v by $x = u + v$ and $y = u - v$, we will cover the *same* square with $\frac{-1}{2} < u < \frac{1}{2}$ and $\frac{-1}{2} < v < \frac{1}{2}$. Furthermore, the Jacobian of this transformation equals to 2.

Solution: $\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} (u-v)^2 \, du \, dv = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{u^3}{3} - 2\frac{u^2}{2} v + uv \right]_{u=-\frac{1}{2}}^{\frac{1}{2}} dv = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{12} + v^2) \, dv = 2(\frac{1}{12} + \frac{1}{12}) = \frac{1}{3}$. ■

An important **special case** is integrating a *constant*, say c, which can often be done geometrically. i.e.

$$\iint\limits_{\mathcal{R}} c\,dx\,dy = c \cdot Area(\mathcal{R})$$

▶Applications◀

of two-dimensional integrals to geometry and physics:

▷ An **area**

of a 2-D region $\mathcal{R}$ is computed by

$$\iint\limits_{\mathcal{R}} dx\,dy$$

▷ **Center of mass**

of a 2-D object (LAMINA) is computed by

$$\frac{\iint x\rho(x,y)\,dx\,dy}{\iint \rho(x,y)\,dx\,dy}$$

[x component] and

$$\frac{\iint y\rho(x,y)\,dx\,dy}{\iint \rho(x,y)\,dx\,dy}$$

[y component], where $\rho(x,y)$ is the corresponding *mass density*. When the object is of uniform density ($\rho \equiv const.$), the formulas simplify to

$$\frac{\iint x\,dx\,dy}{\iint dx\,dy}$$

and

$$\frac{\iint y\,dx\,dy}{\iint dx\,dy}$$

▷ **Moment of inertia**

with respect to some axis (this is needed when computing angular acceleration as torque/moment-of-inertia):

$$\iint d(x,y)^2 \cdot \rho(x,y)\,dx\,dy$$

where $d(x,y)$ is the (perpendicular) distance of (x,y) from the axis [when the axis is x, $d \equiv y$ and vice versa; when the axis is z, $d = \sqrt{x^2+y^2}$].

▷ **3-D volume**

$$\iint h(x,y)\,dx\,dy$$

where $h(x,y)$ is the object's 'thickness' (height) at (x,y).

EXAMPLES:

1. Find the center of mass of a half disk of radius R and uniform mass density.

   Solution: We position the object in the upper half plane with its center at the origin, and use polar coordinates to evaluate: $\frac{\int_0^\pi\int_0^R r\sin\varphi\cdot r\,dr\,d\varphi}{\int_0^\pi\int_0^R r\,dr\,d\varphi} = \frac{\frac{R^3}{3}\cdot(\cos 0-\cos\pi)}{\frac{R^2}{2}\cdot\pi} = \frac{4R}{3\pi} = 0.42441R$ [its y component]. From symmetry, its x component must be equal to zero.

2. Find the volume of a cone with circular base of radius R and height H.

   We do this in polar coordinates where the formula for $h(r,\varphi)$ simplifies to $H\cdot\frac{R-r}{R}$.

   Answer: $\frac{H}{R}\int_0^{2\pi}\int_0^R (R-r)\cdot r\,dr\,d\varphi = \frac{2\pi H}{R}\cdot[R\frac{r^2}{2}-\frac{r^3}{3}]_{r=0}^R = \frac{\pi R^2 H}{3}$ (check).

3. Find the volume of a sphere of radius R.

   Solution: Introducing polar coordinates in x,y, the z-thickness is $h(x,y) = 2\sqrt{R^2-r^2}$ [Pythagoras]. Integrating this over the sphere's x,y projection (a circle of radius R) yields $2\int_0^{2\pi}\int_0^R\sqrt{R^2-r^2}\cdot r\,dr\,d\varphi = 4\pi\left[-\frac{1}{3}(R^2-r^2)^{\frac{3}{2}}\right]_{r=0}^R = \frac{4}{3}\pi R^3$ (check).

4. Find the volume of the (solid) cylinder $x^2+z^2<1$ cut along $y=0$ and $z=y$ [i.e. $0<y<z$].

   Solution: Its x,z projection is a *half*-circle $x^2+z^2<1$ with $z>0$, its thickness along y is $h(x,z)=z$. Replacing x and z by polar coordinates, we can readily integrate $\int_0^\pi\int_0^R r\sin\varphi\cdot r\,dr\,d\varphi = \frac{1}{3}\cdot[-\cos\varphi]_{\varphi=0}^\pi = \frac{2}{3}$. There are two alternate ways of computing the volume, integrating the z-thickness over the (x,y) projection, or the x-thickness over $dy\,dz$ [try both of them].

5. Find the volume of the 3-D region defined by $x^2+y^2<1$ and $y^2+z^2<1$ [the common part of two cylinders crossing each other at the right angle].

   Solution: The (x,y) projection of the region is describe by $x^2+y^2<1$ (now a *circle*, not a cylinder), the corresponding z-thickness is $h(x,y)=2\sqrt{1-y^2}$.

   Answer: $2\int_0^{2\pi}\int_0^1\sqrt{1-r^2\sin^2\varphi}\cdot r\,dr\,d\varphi = \int_0^{2\pi}-\frac{2}{3\sin^2\varphi}\left[(1-r^2\sin^2\varphi)^{\frac{3}{2}}\right]_{r=0}^1 d\varphi = \int_0^{2\pi}\frac{2(1-|\cos\varphi|^3)}{3\sin^2\varphi}\,d\varphi = \int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{2(1-|\cos\varphi|^3)}{3\sin^2\varphi}\,d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{2(1-\cos^3\varphi)}{3\sin^2\varphi}\,d\varphi + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\frac{2(1+\cos^3\varphi)}{3\sin^2\varphi}\,d\varphi = \frac{16}{3}$. [The integration is quite tricky, later on we learn how to deal with it more efficiently].

An alternate way is to use the (x,z) projection (a unit square, divided by its two diagonals into four sections of identical volume), and then integrate over one of these sections (say the right-most) the corresponding y-thickness $h(x,z) = 2\sqrt{1-x^2}$, thus: $2\int_0^1 \sqrt{1-x^2}\int_{-x}^{x} dz\,dx = 2\int_0^1 \sqrt{1-x^2}\cdot 2x\,dx = -\frac{4}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_{x=0}^{1} = \frac{4}{3}$. The total volume is four times bigger (check). The integration was now a lot easier.

In these type of questions, it is important to first identify each SIDE of the 3-D object (and the corresponding equation), and each of its EDGES (described by two equations). To **project** a specific edge into, say, the (x,y) plane, one must eliminate z from one of the two equations and substitute into the other (getting a single x-y equation).

Surfaces in 3-D

There are two ways of **defining** a 2-D surface:

1. By an **equation**: $f(x,y,z) = c$ [c being a constant].

2. **Parametrically**: $\mathbf{r}(u,v) \equiv [x(u,v), y(u,v), z(u,v)]$ (three arbitrary functions of two parameters u and v; restricting these to a 2-D region selects a SECTION of the surface). ■

EXAMPLES:

- Parametrize a *sphere* of radius a.

  Answer: $\mathbf{r}(u,v) = [a\sin v\cos u, a\sin v\sin u, a\cos v]$ where $0 \le u < 2\pi$ and $0 \le v \le \pi$ [later on we introduce the so called SPHERICAL COORDINATES in almost the same manner – they are usually called r, θ and φ rather than a, v and u]. The curves we get by fixing v and varying u (or vice versa) are called 'COORDINATE' CURVES [latitude circles and longitude half-circles in this case].

- Identify $\mathbf{r}(u,v) = [u\cos v, u\sin v, u]$.

  Answer: a 45^o *cone* centered on z.

- Parametrize the *cylinder* $x^2 + y^2 = a^2$.

  Solution: $\mathbf{r}(u,v) = [a\cos u, a\sin u, v]$.

- Identify $[u\cos v, u\sin v, u^2]$.

  Answer: A *paraboloid* centered on $+z$.

Surface integrals

Let us consider a **specific parametrization** of a surface. It is obvious that $\frac{\partial \mathbf{r}}{\partial u}$ [componentwise operation, keeping v fixed] is a *tangent direction* to the corresponding coordinate curve and consequently tangent to the surface itself. Similarly, so is $\frac{\partial \mathbf{r}}{\partial v}$ (note that these two don't have to be orthogonal). Constructing the corresponding TANGENT PLANE is then quite trivial.

Consequently,

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

yields a direction *normal* (perpendicular) to the surface, and its magnitude $\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|$, multiplied by $du\,dv$, provides the *area* of the corresponding (infinitesimal) *parallelogram*, obtained by increasing u by du and v by dv [$\frac{\partial \mathbf{r}}{\partial u}du$ and $\frac{\partial \mathbf{r}}{\partial v}dv$ being its two sides]. This can be seen from:

$$\left|\frac{\partial \mathbf{r}}{\partial u}du \times \frac{\partial \mathbf{r}}{\partial v}dv\right| = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial u}\right| du\,dv \equiv dA$$

Since $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2\gamma = |\mathbf{a}|^2\,|\mathbf{b}|^2\,(1-\cos^2\gamma) = |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a} \bullet \mathbf{b})^2$, we can simplify it to

$$dA = \sqrt{\left|\frac{\partial \mathbf{r}}{\partial u}\right|^2 \left|\frac{\partial \mathbf{r}}{\partial v}\right|^2 - \left(\frac{\partial \mathbf{r}}{\partial u} \bullet \frac{\partial \mathbf{r}}{\partial v}\right)^2}\, du\,dv$$

which is more convenient computationally (bypassing the cross product).

To find an **area** of a whole **surface** (or its section), we need to 'add' the contributions from all these parallelograms, thus:

$$Area = \iint_{\mathcal{S}} dA = \iint_{\mathcal{R}} \sqrt{\left|\frac{\partial \mathbf{r}}{\partial u}\right|^2 \left|\frac{\partial \mathbf{r}}{\partial v}\right|^2 - \left(\frac{\partial \mathbf{r}}{\partial u} \bullet \frac{\partial \mathbf{r}}{\partial v}\right)^2}\, du\,dv$$

where $\mathcal{R}$ is the (u, v) region needed to cover the (section of the) surface $\mathcal{S}$. Needless to say, the answer must be the same, *regardless* of the parametrization.

EXAMPLES:

1. Find the tangent plane to the *ellipsoid* $3x^2 + 2y^2 + z^2 = 20$ at $(1, 2, 3)$.

   Solution: First one can easily check that the point is on the ellipsoid (just in case). We can parametrize the *upper half* of the ellipsoid (which is sufficient in this case) by $\mathbf{r}(u, v) = (u, v, \sqrt{20 - 3u^2 - 2v^2})$. Then $\frac{\partial \mathbf{r}}{\partial u} = (1, 0, -\frac{3u}{\sqrt{20-3u^2-2v^2}}) = (1, 0, -1)$ and $\frac{\partial \mathbf{r}}{\partial v} = (0, 1, -\frac{2v}{\sqrt{20-3u^2-2v^2}}) = (0, 1, -\frac{4}{3})$. The corresponding cross product $(1, 0, -1) \times (0, 1, -\frac{4}{3}) = (1, \frac{4}{3}, 1)$ yields the tangent plane's normal; we also know that the plane has to pass through $(1, 2, 3)$.

   Answer: $3x + 4y + 3z = 20$.

2. Find the area of a surface of a *sphere* of radius a.

   Solution: Using the $\mathbf{r}(u,v) = (a\sin v\cos u,\ a\sin v\sin u,\ a\cos v)$ parametrization, we get: $\frac{\partial \mathbf{r}}{\partial u} = (-a\sin v\sin u,\ a\sin v\cos u,\ 0)$ and $\frac{\partial \mathbf{r}}{\partial v} = (a\cos v\cos u,\ a\cos v\sin u,\ -a\sin v) \Rightarrow dA \equiv a^2\,|\sin v|\,du\,dv$.

   Answer: $a^2\int\limits_0^{2\pi}\int\limits_0^{\pi}\sin v\,dv\,du = 4\pi a^2$.

3. Find the surface area of a TORUS (donut) of dough-radius equal to b and hole-radius equal to $a-b$.

   Solution: We make z its axis, and $[0, a + b\cos v, b\sin v]$ its cross section with the (y,z) plane. The full parametrization is then: $\mathbf{r}(u,v) = [(a + b\cos v)\cos u,\ (a+b\cos v)\sin u,\ b\sin v]$, where both u and v vary from 0 to 2π. This yields $\frac{\partial \mathbf{r}}{\partial u} = [-(a+b\cos v)\sin u,\ (a+b\cos v)\cos u, 0]$, $\frac{\partial \mathbf{r}}{\partial v} = [-b\sin v\cos u,\ b\sin v\sin u,\ -b\cos v] \Rightarrow dA \equiv b(a+b\cos v)$.

   Answer: $b\int\limits_0^{2\pi}\int\limits_0^{2\pi}(a+\cos v)\,dv\,du = 4\pi^2 ab$.

Computing areas is just a special case of a

►Surface Integral of Type I◄

('SCALAR' type). In general, we can integrate any scalar function $f(x,y,z)$ over a surface $\mathcal{S}$ [symbolic **notation** $\iint\limits_{\mathcal{S}} f(x,y,z)\,dA$] by parametrizing the surface and computing

$$\iint\limits_{\mathcal{R}} f[\mathbf{r}(u,v)]\cdot\sqrt{\left|\frac{\partial \mathbf{r}}{\partial u}\right|^2\left|\frac{\partial \mathbf{r}}{\partial v}\right|^2 - \left(\frac{\partial \mathbf{r}}{\partial u}\bullet\frac{\partial \mathbf{r}}{\partial v}\right)^2}\,du\,dv$$

[the answer is independent of parametrization].

When divided by the corresponding surface *area*, this represents the **average** of $f(x,y,z)$ over $\mathcal{S}$.

Other applications to Physics are:

1. **Moment of inertia** of a shell-like structure (LAMINA) of surface *density* $\rho(x,y,z)$:

   $$\iint\limits_{\mathcal{S}} d^2\cdot\rho\cdot dA$$

   where $d(x,y,z)$ is the distance from the rotation axis. For a lamina of *uniform* density, $\rho = \frac{M}{A}$ (total mass over total area).

2. **Center of mass**

   $$\left[\frac{\iint\limits_{\mathcal{S}} x\cdot\rho\cdot dA}{\iint\limits_{\mathcal{S}}\rho\cdot dA}, \frac{\iint\limits_{\mathcal{S}} y\cdot\rho\cdot dA}{\iint\limits_{\mathcal{S}}\rho\cdot dA}, \frac{\iint\limits_{\mathcal{S}} z\cdot\rho\cdot dA}{\iint\limits_{\mathcal{S}}\rho\cdot dA}\right]$$

   (ρ cancels out when constant, i.e. uniform mass density). Note that $\iint\limits_{\mathcal{S}}\rho\cdot dA$ is the total mass. □

EXAMPLE: Find the moment of inertia of a spherical shell of radius a and total mass M (uniformly distributed) with respect to an axis going through its center.

Solution: 'Borrowing' the parametrization (and dA) from the previous Example 2, and using z as the axis, we get $\rho \iint\limits_{\mathcal{S}} (x^2+y^2)\,dA = \rho \int\limits_0^{\pi}\int\limits_0^{2\pi} a^2 \sin^2 v \cdot a^2 \sin v \, du\, dv = 2\pi\rho a^4 [\frac{\cos^3 v}{3} - \cos v]_{v=0}^{\pi} = 2\pi \frac{M}{4\pi a^2} a^4 \cdot \frac{4}{3} = \frac{2}{3} M a^2$. ■

▶Surface integrals of Type II◀

('VECTOR' type): When integrating a vector field $\mathbf{g}(x,y,z)$ [representing some stationary flow] over an ORIENTABLE (having two sides) surface $\mathcal{S}$, we are usually interested in computing the **total flow** (flux) through this surface, in a chosen direction.

The flow through an '**infinitesimal**' area [our parallelogram] of the surface is given by the dot product

$$\mathbf{g} \bullet \mathbf{n}\, dA$$

where $\mathbf{n}$ is a unit direction *normal* [perpendicular] to the area, since the flow is obviously proportional to the area's size dA, to the magnitude of $\mathbf{g}$ (the flow's speed), *and* to the cosine of the $\mathbf{n}$-$\mathbf{g}$ angle.

'Adding' these, one gets

$$\iint\limits_{\mathcal{S}} \mathbf{g} \bullet \mathbf{n}\, dA \equiv \iint\limits_{\mathcal{S}} \mathbf{g} \bullet d\mathbf{A} \equiv \iint\limits_{\mathcal{S}} (g_1\, dy\, dz + g_2\, dz\, dx + g_3\, dx\, dy)$$

introducing two more alternate, **symbolic notation**s (I usually use the middle one).

We can convert this to a regular **double-integral** (in u and v), by *parametrizing* the surface [different parametrizations must give the same correct answer] and replacing $\mathbf{n}\, dA$ by $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ [having both the correct *area* and *direction*], getting:

$$\iint\limits_{\mathcal{R}} \mathbf{g}[\mathbf{r}(u,v)] \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right] du\, dv$$

where $\mathcal{R}$ is the (u,v) region corresponding to $\mathcal{S}$. Note that $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ does not necessarily have the correct (originally prescribed) *orientation*; when that happens, we fix it by reversing the sign of the result.

EXAMPLES (to simplify our notation, we use $\frac{\partial \mathbf{r}}{\partial u} \equiv \mathbf{r}_u$ and $\frac{\partial \mathbf{r}}{\partial v} \equiv \mathbf{r}_v$):

1. Evaluate $\iint\limits_{\mathcal{S}} (x, y, z-3) \bullet d\mathbf{A}$ where $\mathcal{S}$ is the upper (i.e. $z > 0$) half of the $x^2 + y^2 + z^2 = 9$ sphere, oriented upwards.

   Solution: Here we can bypass spherical coordinates (why?) and use instead $\mathbf{r}(u,v) = \left[u, v, \sqrt{9-u^2-v^2}\right]$ with $u^2 + v^2 < 9$ [defining the two-dimensional region $\mathcal{R}$ over which we integrate]. Furthermore, $\mathbf{r}_u = [1, 0, -\frac{u}{\sqrt{9-u^2-v^2}}]$ and

$\mathbf{r}_v = [0, 1, -\frac{v}{\sqrt{9-u^2-v^2}}] \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = [\frac{u}{\sqrt{9-u^2-v^2}}, \frac{v}{\sqrt{9-u^2-v^2}}, 1]$ [correct orientation!] $\Rightarrow \mathbf{g} \bullet (\mathbf{r}_u \times \mathbf{r}_v) = \frac{u^2+v^2}{\sqrt{9-u^2-v^2}} + \sqrt{9-u^2-v^2} - 3 = \frac{9}{\sqrt{9-u^2-v^2}} - 3$. The actual integration will be done in polar coordinates: $\int\limits_0^{2\pi}\int\limits_0^{3}\left(\frac{9}{\sqrt{9-r^2}} - 3\right) r\,dr\,d\varphi =$

$2\pi\left[-9\sqrt{9-r^2} - 3\frac{r^2}{2}\right]_{r=0}^{3} = 27\pi.$

2. Evaluate $\iint\limits_{\mathcal{S}} (yz, xz, xy) \bullet d\mathbf{A}$ where $\mathcal{S}$ is the full $x^2+y^2+z^2 = 1$ sphere oriented outwards. Using the usual parametrization: $\mathbf{r}(u, v) = (\cos u \sin v,\ \sin u \sin v,\ \cos v)$ $\Rightarrow \mathbf{r}_u = (-\sin u \sin v, \cos u \sin v, 0)$, $\mathbf{r}_v = (\cos u \cos v, \sin u \cos v, -\sin v)$ and $\mathbf{r}_u \times \mathbf{r}_v = (-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v)$ [wrong orientation, reverse its sign!], we get $\mathbf{g} \bullet (-\mathbf{r}_u \times \mathbf{r}_v) = 3\cos u\ \sin u\ \sin^3 v\ \cos v$.

   Answer: $3\int\limits_0^{2\pi} \sin u \cos u\, du \times \int\limits_0^{\pi} \sin^3 v\ \cos v\, dv = 0$ ■

Shortly we learn a shortcut for evaluating Type II integrals over a *closed* surface which will make the last example trivial. But first we need to discuss

'Volume' integrals

which are only of the *scalar* type (there is no natural direction to associate with an infinitesimal volume, say a cube; contrast this with the tangent direction for a curve and the normal direction for a surface).

The other difference is that the 3-D integration can be carried out directly in terms of x, y and z, which are in a sense direct 'parameters' of the corresponding 3-D region (sometimes called 'volume'). This is not to say that we can not try different (more convenient) ways of 'parametrizing' it, but this will now be referred to as a 'CHANGE OF VARIABLES' (or introducing generalized coordinates).

The most typical example of these are the **spherical coordinates** r, θ and φ (to simplify integrating over a sphere). When using spherical coordinates, $dx\,dy\,dz$ ($\equiv dV$) needs to be replaced by $dr\,d\theta\,d\varphi$ *multiplied* by the JACOBIAN of the transformation, namely: $\begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} =$

$$r^2\sin\theta$$

[this expression can be derived and understood geometrically].

Similarly to double integration of a **constant**, some triple integrals can be also evaluated 'geometrically', by

$$\iiint\limits_{\mathcal{V}} c\,dV = c \cdot Volume(\mathcal{V})$$

whenever the 3-D region is of a simple enough shape, and we remember a formula for the corresponding volume.

▶Possible Applications◀

of volume integrals include computing the *actual* **volume** of a 3-D body

$$V = \iiint_{\mathcal{V}} dV$$

averaging a scalar function $f(x, y, z)$ over a 3-D region

$$\frac{\iiint_{\mathcal{V}} f(x, y, z)\, dV}{V}$$

computing the **center of mass** of a 3-D object of mass density $\rho(x, y, z)$ [it cancels out when constant]

$$\left[\frac{\iiint_{\mathcal{V}} x\, \rho(x, y, z)\, dV}{\iiint_{\mathcal{V}} \rho(x, y, z)\, dV}, \frac{\iiint_{\mathcal{V}} y\, \rho(x, y, z)\, dV}{\iiint_{\mathcal{V}} \rho(x, y, z)\, dV}, \frac{\iiint_{\mathcal{V}} z\, \rho(x, y, z)\, dV}{\iiint_{\mathcal{V}} \rho(x, y, z)\, dV} \right]$$

and computing the corresponding **moment of inertia**

$$\iiint_{\mathcal{V}} d^2 \rho\, dV$$

where $d(x, y, z)$ is distance from the rotational axis, and $\rho \equiv \frac{M}{V}$ when the mass density is uniform.

EXAMPLE: Find the moment of inertia of a uniform *sphere* of radius a with an axis going through its center.

Solution: $\frac{M}{\frac{4}{3}\pi a^3} \iiint_{\mathcal{V}} (x^2 + y^2) dV = \frac{M}{\frac{4}{3}\pi a^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^2 \sin^2\theta \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi = \frac{M}{\frac{4}{3}\pi a^3} \cdot \frac{a^5}{5} \cdot \left[\frac{\cos^3\theta}{3} - \cos\theta \right]_{\theta=0}^{\pi} \cdot 2\pi = \frac{2}{5} M a^2$. ■

There is an interesting and **useful relationship** between a Type II integral over a closed (outward oriented) surface $\mathcal{S}_c$, and a volume integral over the 3-D region $\mathcal{V}$ enclosed by this $\mathcal{S}_c$, called

▶Gauss Theorem◀

$$\iint_{\mathcal{S}_c} \mathbf{g} \bullet d\mathbf{A} \equiv \iiint_{\mathcal{V}} Div(\mathbf{g})\, dV$$

[$Div(\mathbf{g})$ must have no singularities throughout $\mathcal{V}$].

Indication of Proof: We have already seen this to be true for an infinitesimal volume $dx\, dy\, dz$ when we introduced divergence of a vector field. When the contributions of all these infinitesimal volumes are added together (to build the surface integral on the left hand side of our formula) the adjacent-side flows cancel out and we are left with the overall surface only; adding the divergences (each multiplied by the corresponding infinitesimal volume) results in the right-hand-side integral. □

EXAMPLES:

1. The integral of Example 2 from the previous section thus becomes quite trivial, as $Div\,([yz, xz, xy]) \equiv 0$.

2. Evaluate $\iint\limits_{\mathcal{S}} (x^3, x^2y, x^2z) \bullet \mathbf{n}\, dA$, where $\mathcal{S}$ is the surface of $\left\{ \begin{array}{c} x^2+y^2<a^2 \\ 0<z<b \end{array} \right.$ (a cylinder of radius a and height b), oriented outwards.

   Solution: Using the Gauss theorem, we get $\iiint\limits_{\substack{x^2+y^2<a^2 \\ 0<z<b}} 5x^2\, dV = 5b \iint\limits_{x^2+y^2<a^2} x^2\, dx\, dy =$ $5b \int\limits_0^{2\pi}\int\limits_0^{a} r^2 \cos^2\varphi \cdot r\, dr\, d\varphi$ [going polar] $= 5b^2 \cdot \frac{a^4}{4} \cdot \pi = \frac{5}{4}a^4 b\pi$.

   Let us *verify* this by recomputing the original surface integral directly (note that now we have to deal with three distinct surfaces: the top disk, the bottom disk, and the actual cylindrical walls): The top can be parametrized by $\mathbf{r}(u,v) = [u, v, b]$, contributing $\iint\limits_{u^2+v^2<a^2} [u^3, u^2v, u^2b] \bullet (0,0,1)\, du\, dv = b \int\limits_0^{2\pi}\int\limits_0^{a} r^2\cos^2\varphi \cdot$ $r\, dr\, d\varphi$ [polar] $= b\frac{a^4}{4}\pi$. The bottom is parametrized by $\mathbf{r}(u,v) = [u,v,0]$, contributing minus (because of the wrong orientation) $\iint\limits_{u^2+v^2<a^2} [u^3, u^2v, 0] \bullet$ $(0,0,1)\, du\, dv \equiv 0$. Finally, the sides are parametrized by $\mathbf{r}(u,v) = [a\cos u, a\sin u, v]$, contributing $\iint\limits_{\substack{0<u<2\pi \\ 0<v<b}} [a^3\cos^3 u, a^3\cos^2 u \sin u, a^2 v\cos^2 u] \bullet [a\cos u, a\sin u, 0]\, du\, dv =$ $a^4 \int\limits_0^{b}\int\limits_0^{2\pi} \cos^2 u\, du\, dv = a^4 b\pi$. Adding the three contributions gives $\frac{5}{4}a^4b\pi$ [check].

   ■

Similarly, there is an **interesting relationship** between the Type II *line* integral over a *closed* curve $\mathcal{C}_d$ and a Type II *surface* integral over *any* surface $\mathcal{S}$ having $\mathcal{C}_d$ as its boundary, called

►Stokes' Theorem◄

$$\iint\limits_{\mathcal{S}} Curl(\mathbf{g}) \bullet \mathbf{n}\, dA \equiv \oint\limits_{\mathcal{C}_d} \mathbf{g} \bullet d\mathbf{r}$$

where the orientation of $\mathcal{C}_d$ and that of $\mathbf{n}\, dA$ follow the right-handed pattern. [When $\mathcal{C}_d$ and $\mathcal{S}$ lie in the (x,y) plane, this is known as the Green's Theorem].

Indication of Proof (which is, this time, a lot more complicated):

We parametrize $\mathcal{S}$ and then, using the corresponding coordinate lines, we divide $\mathcal{S}$ into many infinitesimal parallelograms and evaluate $\oint \mathbf{g} \bullet d\mathbf{r}$ for each of these. When adding these together, the contributions of any two adjacent sides cancel out, and we end up with the integral on the right hand side of the Stokes' formula.

On the other hand, the contribution of each of these integrals can be approximated (the approximation becomes exact in the appropriate limit) by the difference in $\mathbf{g}$ between two opposite sides of the parallelogram ($\frac{\partial \mathbf{g}}{\partial u}\,du$ and $\frac{\partial \mathbf{g}}{dv}\,dv$) dot-multiplied by the vector representation of the two sides ($\frac{\partial \mathbf{r}}{\partial v}\,dv$ and $\frac{\partial \mathbf{r}}{\partial u}\,du$, respectively). These two are then subtracted (since one runs with, and the other one against, the counterclockwise orientation of the boundary) to get

$$\oint \mathbf{g}\bullet d\mathbf{r} \simeq \left(\frac{\partial \mathbf{g}}{\partial u}\bullet\frac{\partial \mathbf{r}}{\partial v} - \frac{\partial \mathbf{g}}{\partial v}\bullet\frac{\partial \mathbf{r}}{\partial u}\right) du\,dv$$

Using the chain rule for expanding $\frac{\partial \mathbf{r}}{\partial v}$ and $\frac{\partial \mathbf{r}}{\partial u}$, and expressing the dot products in terms of individual components, we get $\sum\limits_{i,j=1}^{3}\left(\frac{\partial g_j}{\partial r_i}\cdot\frac{\partial r_i}{\partial u}\cdot\frac{\partial r_j}{\partial v} - \frac{\partial g_j}{\partial r_i}\cdot\frac{\partial r_i}{\partial v}\cdot\frac{\partial r_j}{\partial u}\right) du\,dv = \sum\limits_{i,j,k,\ell=1}^{3}\frac{\partial g_j}{\partial r_i}\cdot\frac{\partial r_k}{\partial u}\cdot\frac{\partial r_\ell}{\partial v}\left(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk}\right) = \sum\limits_{i,j,k,\ell,m=1}^{3}\frac{\partial g_j}{\partial r_i}\cdot\frac{\partial r_k}{\partial u}\cdot\frac{\partial r_\ell}{\partial v}\cdot\epsilon_{ijm}\epsilon_{k\ell m} = Curl(\mathbf{g})\bullet\left(\frac{\partial \mathbf{r}}{\partial v}\times\frac{\partial \mathbf{r}}{\partial u}\right)$. These, when added together, result in the left hand side of our formula. □

EXAMPLE: Evaluate $\oint\limits_{\mathcal{C}_d}(y, xz^3, -zy^3)\bullet d\mathbf{r}$, where $\mathcal{C}_d$ is defined by $\left\{\begin{array}{c} x^2+y^2=4 \\ z=-3\end{array}\right.$, counterclockwise when viewed from the top.

Solution: Using Stokes' Theorem we replace this integral by $\iint\limits_{\mathcal{S}}[-3zy^2 - 3xz^2,\ 0,\ z^3-1]\bullet\mathbf{n}\,dA$, where $\mathcal{S}$ is the corresponding (flat) disk. Parametrizing $\mathcal{S}$ by $\mathbf{r}(u,v)\equiv[u,v,-3] \Rightarrow \mathbf{n}\,dA=[0,0,1]\,du\,dv$, this converts to $\iint\limits_{u^2+v^2<4}(-28)\,du\,dv = -28\cdot 4\pi = -112\pi$ [note that we did not need to know the first two components of $Curl(\mathbf{g})$ in this case, i.e. it pays to do the $\mathbf{n}\,dA$ *first*].

We will verify the answer by performing the original line integral, directly: $\mathbf{r}(t) = [2\cos t, 2\sin t, -3]$ is the parametrization of $\mathcal{C}_d$, which converts the integral to $\int\limits_0^{2\pi}[2\sin t, -54\cos t, 24\sin^3 t]\bullet[-2\sin t, 2\cos t, 0]\,dt = \int\limits_0^{2\pi}(-4\sin^2 t - 108\cos^2 t)\,dt = -112\pi$ [almost equally easily]. ■

Unless $Curl(\mathbf{g})\equiv\mathbf{0}$, the computational simplification achieved by applying the Stokes' theorem is very limited (a far cry from the Gauss theorem). One exception is when $\mathcal{C}_d$ is a 'broken' planar curve (consisting of several segments), as we can trade *one* surface integral for *several* line integrals.

Review exercises

1. Find the area of the following (truncated) paraboloid: $\left\{\begin{array}{c} z=x^2+y^2 \\ z<b\end{array}\right.$.

Solution: Parametrize: $\mathbf{r}=[u,v,u^2+v^2] \Rightarrow \mathbf{r}_u=[1,0,2u]$ and $\mathbf{r}_v=[0,1,2v] \Rightarrow dA=\sqrt{(1+4u^2)(1+4v^2)-16u^2v^2}\,du\,dv = \sqrt{1+4(u^2+v^2)}\,du\,dv$. We need $\iint\limits_{u^2+v^2<b} dA =$ [going polar] $\int\limits_0^{2\pi}\int\limits_0^{\sqrt{b}}\sqrt{1+4r^2}\cdot r\,dr\,d\varphi = 2\pi\cdot\left[\frac{1}{12}(1+4r^2)^{\frac{3}{2}}\right]_{r=0}^{\sqrt{b}} = \frac{\pi}{6}\left[(1+4b)^{\frac{3}{2}}-1\right]$.

2. Evaluate $\iint\limits_{\mathcal{S}} [y, 2, xz] \bullet \mathbf{n}\, dA$, where $\mathcal{S}$ is defined by $\begin{cases} y = x^2 \\ 0 < x < 2 \\ 0 < z < 3 \end{cases}$, and $\mathbf{n}$ is pointing in the direction of $-y$.

   Solution: $\mathbf{r}(u, v) = [u, u^2, v] \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = [1, 2u, 0] \times [0, 0, 1] = [2u, -1, 0]$ (correct orientation). The integral thus converts to $\int\limits_0^3 \int\limits_0^2 [u^2, 2, uv] \bullet [2u, -1, 0]\, du\, dv = \int\limits_0^3 \int\limits_0^2 (2u^3 - 2)\, du\, dv = 3\left[\frac{u^4}{2} - 2u\right]_{u=0}^2 = 12.$

3. Find $\iint\limits_{\mathcal{S}} [x^2, 0, 3y^2] \bullet \mathbf{n}\, dA$, where $\mathcal{S}$ is the $\begin{cases} x > 0 \\ y > 0 \\ z > 0 \end{cases}$ portion of the $x+y+z = 1$ plane, and $\mathbf{n}$ is pointing upwards.

   Solution: $\mathbf{r} = [u, v, 1-u-v] \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = [1, 0, -1] \times [0, 1, -1] = [1, 1, 1]$ (correct orientation) $\Rightarrow \int\limits_0^1 \int\limits_0^{1-v} [u^2, 0, 3v^2] \bullet [1, 1, 1]\, du\, dv = \int\limits_0^1 \int\limits_0^{1-v} (u^2 + 3v^2)\, du\, dv = \int\limits_0^1 \left[\frac{u^3}{3} + 3uv^2\right]_{u=0}^{1-v} dv = \int\limits_0^1 \left[\frac{(1-v)^3}{3} + 3(1-v)v^2\right] du = \left[-\frac{(1-v)^4}{12} + 3\frac{v^3}{3} - 3\frac{v^4}{4}\right]_{v=0}^1 = \frac{1}{3}.$

4. Parametrize a circle of radius $\rho = 5$, centered on $\mathbf{a} = [1, -2, 4]$, and normal to $\mathbf{n} = [2, 0, -3]$.

   Solution: In general, a circle is parametrized by: $\mathbf{r}(t) = \mathbf{a} + \rho \mathbf{m}_1 \cos t + \rho \mathbf{m}_2 \sin t$, where $\mathbf{m}_1$ and $\mathbf{m}_2$ are *unit* vectors perpendicular to $\mathbf{n}$ and to each other. They can be found by taking the cross product of $\mathbf{n}$ and an *arbitrary* vector, then taking the cross product of the resulting vector and $\mathbf{n}$, and normalizing both, thus: $[2, 0, -3] \times [1, 0, 0] = [0, -3, 0]$ and $[0, -3, 0] \times [2, 0, -3] = [9, 0, 6] \Rightarrow \mathbf{m}_1 = [0, -3, 0] \div 3 = [0, -1, 0]$ and $\mathbf{m}_2 = [9, 0, 6] \div \sqrt{9^2 + 6^2} = \left[\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}\right]$.

   Answer: $\mathbf{r}(t) = [1 + \frac{15}{\sqrt{13}} \sin t,\ -2 - 5\cos t,\ 4 + \frac{10}{\sqrt{13}} \sin t]$ where $0 \le t < 2\pi$. Subsidiary: To parametrize the corresponding *disk:* $\mathbf{r}(u, v) = [1 + \frac{3v}{\sqrt{13}} \sin u,\ -2 - v\cos u,\ 4 + \frac{2v}{\sqrt{13}} \sin u]$ where $0 \le u < 2\pi$ and $0 \le v < 5$.

5. Find the moment of inertia (with respect to the z axis) of a shell-like *torus* (parametrized earlier) of uniform mass density and total mass M.

   Solution: Recall that $\mathbf{r}(u, v) = [(a + b\cos v)\cos u,\ (a + b\cos v)\sin u,\ b\sin v] \Rightarrow dA = b(a + b\cos v)\ du\, dv$ [done earlier] and $d^2 = (a + b\cos v)^2 \Rightarrow \rho \int\limits_0^{2\pi} \int\limits_0^{2\pi} (a + b\cos v)^2 b(a + b\cos v)\, du\, dv = \rho b 2\pi \int\limits_0^{2\pi} (a^3 + 3a^2 b\cos v + 3ab^2 \cos^2 v + b^3 \cos^3 v)\, dv = \frac{M}{4\pi^2 ab} b 2\pi [2\pi a^3 + 3ab^2 \pi] = M\,(a^2 + \frac{3}{2} b^2).$

6. Repeat with a *solid* torus.

Solution: We replace $\mathbf{r}$ by $[(a+r\cos v)\cos u,\ (a+r\cos v)\sin u,\ r\sin v]$, where the new variables u, v and r $(0 \le r < b)$ can be also seen as *orthogonal* coordinates. For any orthogonal coordinates it is easy to find the Jacobian, *geometrically*, by $dx\,dy\,dz \to r\,dv\cdot(a+r\cos v)\,du\cdot dr = r\,(a+r\cos v)\,du\,dv\,dr$ $\Rightarrow \rho\int_0^b\int_0^{2\pi}\int_0^{2\pi}(a+r\cos v)^2 r\,(a+r\cos v)\,du\,dv\,dr = \rho 2\pi^2\int_0^b r\,(2a^3+3ar^2)\,dr = \rho 2\pi^2(a^3b^2+\frac{3}{4}ab^4)$.

Similarly, the total *volume* is $\int_0^b\int_0^{2\pi}\int_0^{2\pi} r\,(a+r\cos v)\,du\,dv\,dr = (2\pi)^2 a\frac{b^2}{2}$.

Answer: $\frac{M}{2\pi^2 ab^2}2\pi^2(a^3b^2+\frac{3}{4}ab^4) = M\,(a^2+\frac{3}{4}b^2)$.

An alternate approach would introduce polar coordinates in the (x,y)-plane, use $2\sqrt{b^2-(r-a)^2}$ for the z-thickness and r^2 for d^2, leading to $\rho\int_0^{2\pi}\int_{a-b}^{a+b} r^2\cdot 2\sqrt{b^2-(r-a)^2}\cdot r\,dr\,d\varphi = \ldots$ [verify that this leads to the same answer].

7. Consider the following solid $\begin{cases} x>0 \\ y>0 \\ z>0 \\ x+y+z<1 \end{cases}$ of uniform density. Find:

(a) Center of mass.

Solution: To find its x-component we need to divide $\int_0^1\int_0^{1-z}\int_0^{1-y-z} x\,dx\,dy\,dz = \int_0^1\int_0^{1-z}\frac{(1-y-z)^2}{2}dy\,dz = \int_0^1\frac{(1-z)^3}{6}dz = \frac{1}{24}$ by the volume $\int_0^1\int_0^{1-z}\int_0^{1-y-z} dx\,dy\,dz = \int_0^1\int_0^{1-z}(1-y-z)dy\,dz = \int_0^1\frac{(1-z)^2}{2}dz = \frac{1}{6}$.

Answer: $[\frac{1}{4},\frac{1}{4},\frac{1}{4}]$, as the y and z-components must have the same value as the x-component [obvious from symmetry].

(b) Moment of inertial with respect to $[t,t,t]$ (the axis).

Solution: To find d^2 we project $[x,y,z]$ into $[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}]$ (unit direction of the axis), getting $[x,y,z]\bullet[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}] = \frac{x+y+z}{\sqrt{3}}$. By Pythagoras, $d^2 = x^2+y^2+z^2-\left[\frac{x+y+z}{\sqrt{3}}\right]^2$.

Answer: $\frac{M}{V}\int_0^1\int_0^{1-z}\int_0^{1-y-z}\left[x^2+y^2+z^2-\frac{(x+y+z)^2}{3}\right]dx\,dy\,dz =$
$4M\int_0^1\int_0^{1-z}\int_0^{1-y-z}[x^2+y^2+z^2-xy-xz-yz]\,dx\,dy\,dz =$
$12M\int_0^1\int_0^{1-z}\int_0^{1-y-z}[x^2-xz]\,dx\,dy\,dz$ [due to symmetry] $=$

$12M \int\limits_0^1 \int\limits_0^{1-z} \left[\frac{(1-y-z)^3}{3} - \frac{(1-y-z)^2}{2} z \right] dy\, dz = M \int\limits_0^1 [(1-z)^4 - 2(1-z)^3 z]\, dz =$
$M \left[-\frac{(1-z)^5}{5} + 2\frac{(1-z)^4}{4} z + 2\frac{(1-z)^5}{20} \right]_0^1 = \frac{M}{10}.$

8. A container is made of a spherical shell of radius 1 and height h. Find:

   (a) The shell's surface area.

   Solution: $\mathbf{r}(u,v) = [u, v, -\sqrt{1-u^2-v^2}] \Rightarrow \mathbf{r}_u = [1, 0, \frac{u}{\sqrt{1-u^2-v^2}}]$ and $\mathbf{r}_v = [0, 1, \frac{v}{\sqrt{1-u^2-v^2}}] \Rightarrow dA = \sqrt{\left(1 + \frac{u^2}{1-u^2-v^2}\right)\left(1 + \frac{v^2}{1-u^2-v^2}\right) - \frac{u^2v^2}{(1-u^2-v^2)^2}}\, du\, dv \equiv \sqrt{\frac{1}{1-u^2-v^2}}\, du\, dv.$

   Answer: $\iint\limits_{u^2+v^2<h(2-h)} \frac{du\, dv}{\sqrt{1-u^2-v^2}} = \int\limits_0^{2\pi} \int\limits_0^{\sqrt{h(2-h)}} \frac{r\, dr}{\sqrt{1-r^2}}\, d\varphi = 2\pi[-\sqrt{1-r^2}]_{r=0}^{\sqrt{h(2-h)}} = 2\pi[1 - \sqrt{1-h(2-h)}] = 2\pi h.$

   (b) The container's volume:

   Solution: Since the z-thickness (depth) equals $\sqrt{1-x^2-y^2} - (1-h)$, all we need is $\iint\limits_{x^2+y^2<h(2-h)} \left[\sqrt{1-x^2-y^2} - (1-h)\right] dx\, dy = \int\limits_0^{2\pi} \int\limits_0^{\sqrt{h(2-h)}} [\sqrt{1-r^2} - 1 + h] \cdot r\, dr\, d\varphi = 2\pi \left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}} - (1-h)\frac{r^2}{2} \right]_{r=0}^{\sqrt{h(2-h)}} = 2\pi[-\frac{1}{3}(1-h)^3 - (1-h)\frac{h(2-h)}{2} + \frac{1}{3}] = \pi h^2 \left(1 - \frac{h}{3}\right).$

9. Evaluate $\oint\limits_{\mathcal{C}} [(x+y)\, dx + (2x-z)\, dy + (y+z)\, dz]$, where $\mathcal{C}$ is the closed curve consisting of three straight-line segments connecting $[2,0,0]$ to $[0,3,0]$, that to $[0,0,6]$, and back to $[2,0,0]$.

   Solution: Applying the Stokes' Theorem, which enables us to trade *three* line integrals (the three segments would require individual parametrization) for *one* surface integral, we first compute $Curl(\mathbf{g}) = [2,0,1]$, then $\mathbf{r}(u,v) = [u, v, 6-3u-2v]$ (note that $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$ is the equation of the corresponding plane) $\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = [1,0,-3] \times [0,1,-2] = [3,2,1]$ which has the *correct* orientation.

   Answer: $\iint\limits_{\substack{u>0\\v>0\\3u+2v<6}} [2,0,1] \bullet [3,2,1]\, du\, dv = 7 \times Area = 7 \times \frac{2\times 3}{2} = 21$

   [Verify by computing the line integral (broken onto three parts) directly].

10. Evaluate $\oint\limits_{\mathcal{C}} [yz\, dx + xz\, dy + xy\, dz]$, where $\mathcal{C}$ is the intersection of $x^2 + 9y^2 = 9$ and $z = 1 + y^2$ oriented counterclockwise when viewed from above (in terms of z).

    Solution: Applying the same Stokes' Theorem, we get $Curl(\mathbf{g}) \equiv [0,0,0]$.

Answer: 0.

We will verify this by evaluating the line integral *directly*: $\mathbf{r}(t) = [3\cos t, \sin t, 1+\sin^2 t]$ parametrizes the curve $(0 < t < 2\pi) \Rightarrow \int_0^{2\pi} [(1+\sin^2 t)\sin t,\ 3(1+\sin^2 t)\cos t,\ 3\sin t\cos t] \bullet [-3\sin t,\ \cos t,\ 2\sin t\cos t]\, dt = \int_0^{2\pi} [-3\sin^2 t(1+\sin^2 t) + 3(1+\sin^2 t)(1-\sin^2 t) + 6\sin^2 t(1-\sin^2 t)]\, dt = \int_0^{2\pi} (3 + 3\sin^2 t - 12\sin^4 t)\, dt = 2\pi \times \left(3 + 3\times\frac{1}{2} - 12\times\frac{3}{8}\right) = 0$ [check].

Note that $\int_0^{2\pi} \sin^{2n} t\, dt = \int_0^{2\pi} \cos^{2n} t\, dt = 2\pi \times \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} \times .... \times \frac{2n-1}{2n}$.

11. In Physics we learned that the gravitational force of a 'solid' (i.e. 3-D) body exerted on a point-like particle at $\mathbf{R} \equiv [X, Y, Z]$ is given by

$$\mu \iiint_{\mathcal{V}} \rho(\mathbf{r}) \frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3}\, dV$$

where μ is a constant, ρ is the body's mass density, and $\mathcal{V}$ is its 'volume' (i.e. 3-D extent). [Here we are integrating a vector field in the *componentwise* (scalar) sense, i.e. these are effectively *three* volume integrals, not one].

Prove that, when the body is *spherical* (of radius a) and ρ is a function of r *only* (placing the coordinate origin at the body's center), this force equals

$$\mu M \cdot \frac{-\mathbf{R}}{|\mathbf{R}|^3}$$

where M is the body's total mass.

Solution: First we notice that $\frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3} \equiv \boldsymbol{\nabla}_{\mathbf{R}} \frac{1}{|\mathbf{r}-\mathbf{R}|}$, where $\boldsymbol{\nabla}_{\mathbf{R}} \equiv [\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}]$. This implies that $\mu \iiint_{\mathcal{V}} \rho(\mathbf{r}) \frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3}\, dV \equiv$

$$\mu \boldsymbol{\nabla}_{\mathbf{R}} \iiint_{\mathcal{V}} \rho(\mathbf{r}) \frac{1}{|\mathbf{r}-\mathbf{R}|}\, dV$$

leading to a lot easier integration (also, now we need *one*, not three integrals). Evaluating $\iiint_{\mathcal{V}} \rho(\mathbf{r}) \frac{1}{|\mathbf{r}-\mathbf{R}|}\, dV$ (the so called gravitational potential) in spherical coordinates yields $\int_0^a \rho(r) \int_0^\pi \int_0^{2\pi} \frac{r^2 \sin\theta \cdot d\varphi\, d\theta\, dr}{\sqrt{r^2+R^2-2Rr\cos\theta}}$ [note that $|\mathbf{r}-\mathbf{R}| = \sqrt{x^2+y^2+(z-R)^2}$, where we have conveniently chosen the direction of $\mathbf{R}$ (instead of the usual z) to correspond to $\theta = 0$]. This further equals $\frac{2\pi}{R} \int_0^a \rho(r) \cdot r \cdot \left[\sqrt{r^2+R^2-2Rr\cos\theta}\right]_{\theta=0}^{\pi} dr = \frac{2\pi}{R} \int_0^a \rho(r) \cdot r \cdot [(R+r)-(R-r)]\, dr = \frac{4\pi}{R} \int_0^a \rho(r)\, r^2\, dr = \frac{M}{R}$.

This proves our assertion, as $\boldsymbol{\nabla}_{\mathbf{R}}\frac{1}{R} = \boldsymbol{\nabla}_{\mathbf{R}}\frac{1}{\sqrt{X^2+Y^2+Z^2}} =$
$\left[\frac{-X}{(X^2+Y^2+Z^2)^{\frac{3}{2}}}, \frac{-Y}{(X^2+Y^2+Z^2)^{\frac{3}{2}}}, \frac{-Z}{(X^2+Y^2+Z^2)^{\frac{3}{2}}}\right] = \frac{-\mathbf{R}}{R^3}$.
Now try to prove the original statement directly (*bypassing* the potential), you should not find it too difficult.

12. Optional: Using Gauss Theorem (somehow indirectly, because of the singularity at $\mathbf{R} = \mathbf{r}$, but this need not concern us here), one can show that

$$\iiint\limits_{\mathcal{V}} Div_{\mathbf{R}}\left(\frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3}\right) dV_{\mathbf{R}} = -4\pi$$

for any $\mathcal{V}$ containing $\mathbf{r}$, and equals 0 otherwise [note that the variable of both the integration, and the divergence operator, is $\mathbf{R}$, whereas $\mathbf{r}$ is considered a fixed parameter]. This implies that $Div_{\mathbf{R}}\left(\frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3}\right)$ [as a function of $\mathbf{R}$] must be equal to zero everywhere except at $\mathbf{R} = \mathbf{r}$, where its value becomes minus infinity (this can be easily verified by direct differentiation). This infinite 'blip' of its value contributes an exact, finite amount of -4π when the function is integrated. We can change this to $+1$ by a simple division, thus:

$$\frac{1}{-4\pi} Div_{\mathbf{R}}\left(\frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3}\right) \equiv \delta^{(3)}(\mathbf{R}-\mathbf{r})$$

defining the so called (3-D) Dirac's delta function. Its basic property is

$$\iiint\limits_{\mathcal{V}} f(\mathbf{R}) \cdot \delta^{(3)}(\mathbf{R}-\mathbf{r})\, dV_{\mathbf{R}} = f(\mathbf{r})$$

We can now understand why

$$\mathbf{F}(\mathbf{R}) = \mu \iiint\limits_{\mathcal{V}} \rho(\mathbf{r}) \frac{\mathbf{r}-\mathbf{R}}{|\mathbf{r}-\mathbf{R}|^3}\, dV$$

(the gravitational force of the previous example) implies

$$Div_{\mathbf{R}}\left(\mathbf{F}(\mathbf{R})\right) = -4\pi\mu\rho(\mathbf{R})$$

When studying partial differential equations, one learns that the last equality also implies the previous one, and there are thus two equivalent ways of expressing the same law of Physics [in the so called integral and differential form, respectively]. This is essential for understanding Maxwell equations and their experimental basis. $\otimes$

Part III

COMPLEX ANALYSIS

Chapter 9 COMPLEX FUNCTIONS – DIFFERENTIATION

Preliminaries

We already know how to add, subtract, multiply [e.g. $(4+3i)(2-5i) = 8+6i-20i+15 = 23-14i$] and divide [e.g. $\frac{4+3i}{2-5i} = \frac{(4+3i)(2+5i)}{(2-5i)(2+5i)} = \frac{-7+26i}{29} = -\frac{7}{29}+\frac{26}{29}i$] complex numbers. In this chapter, we will learn how to evaluate most of the usual functions using a complex argument (getting, in general, a complex answer). We will also investigate the issue of taking a derivative of any such function.

▶Basic Definitions◀

We reserve the letter $z = x + iy$ for a complex number (soon to become a **complex variable**), where $x = Re(z)$ is its REAL PART and $y = Im(z)$ is its (purely) IMAGINARY PART [these are already two simple examples of *functions* of z].

Similarly, $\bar{z} \equiv x - iy$ (some books use z^*) is the complex CONJUGATE of z [yet another function of z]. It is obvious that $\bar{\bar{z}} \equiv z$ and easy to prove

$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$$

Proof: $\overline{(x_1+iy_1)(x_2+iy_2)} = x_1x_2 - y_1y_2 - i(x_1y_2+x_2y_1)$ and $(x_1-iy_1)(x_2+iy_2) = x_1x_2 - y_1y_2 - i(x_1y_2+x_2y_1)$, which agree □

and

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

[proof similar]. Also note that $z\bar{z} = x^2+y^2$.

Geometrically, complex numbers are often represented as points of the x-y plane, leading to their so called **polar representation**:

$$r = |z| \equiv \sqrt{x^2+y^2}$$

[the MAGNITUDE] and

$$\theta = \pm \arctan(y/x)$$

[the ARGUMENT, where the sign is chosen according to the quadrant of θ]. The value of θ is usually chosen from the $(-\pi, \pi]$ interval (the so called PRINCIPAL VALUE of the argument), but it has obviously infinitely many *potential* values [$\theta \pm 2\pi k$, where k is an integer]. Conversely,

$$z = r(\cos\theta + i\sin\theta)$$

Using this representation, one can easily show that the **product** $z_1 \cdot z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + ir_1r_2(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1) =$

$$r_1r_2[\cos(\theta_1+\theta_2) + i\,\sin(\theta_1+\theta_2)]$$

and similarly, the **ratio** $\frac{z_1}{z_2} =$

$$\frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\ \sin(\theta_1 - \theta_2)]$$

The first of these two formulas can be extended to any number of factors, further implying that an **integer power** of a complex number can be computed from

$$z^n = r^n[\cos(n\theta) + i \sin(n\theta)]$$

This also enables us to derive formulas of the following type: $\cos 5\theta + i \sin 5\theta = (\cos\theta + i \sin\theta)^5 = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$.

EXAMPLES: Find the region (of complex plane) which corresponds to:

1. $|z| \le 1$.

   Solution: $\sqrt{x^2+y^2} \le 1$, i.e. the unit disk centered on $(0,0)$.

2. $|z-1| + |z+1| = 3$.

   Solution: Square $|z-1| = 3 - |z+1|$ to get $(z-1)(\bar{z}-1) = 9 - 6|z+1| + (z+1)(\bar{z}+1) \Leftrightarrow 6|z+1| = 9 + 2(z+\bar{z})$. Square again getting: $36[(x+1)^2+y^2] = 81 + 72x + 16x^2 \Leftrightarrow 20x^2 + 36y^2 = 45$ [ellipse centered on $(0,0)$].

3. $0 < Im(\frac{1}{z}) < 1$.

   Solution: Since $Im(\frac{1}{z}) = Im(\frac{x-iy}{x^2+y^2}) = -\frac{y}{x^2+y^2}$, we get $0 < -y < x^2+y^2$, or $y < 0$ *and* $x^2 + (y+\frac{1}{2})^2 > \frac{1}{4}$, i.e. a set of points below the x-axis and outside the disk of radius $\frac{1}{2}$ centered on $(0, -\frac{1}{2})$. ■

Introducing complex functions

Any expression involving z defines a complex function, e.g.. $f(z) = z^2 + 3z$. In general, any such function will have complex values and can be thus expressed in terms of *two* (real) function, the real part of $f(z)$ and its (purely) imaginary part. These are usually called $u(x,y)$ and $v(x,y)$ respectively, each being a function of x and y (real arguments), i.e.

$$f(z) \equiv u(x,y) + i\,v(x,y)$$

EXAMPLE: $f(z) \equiv z^2 + 3z = x^2 + 2ixy - y^2 + 3x + 3iy = (x^2 - y^2 + 3x) + i(2xy + 3y) \equiv u(x,y) + iv(x,y)$. ■

▶Derivative◀

of a complex function is a fairly difficult concept, even though, its **definition** is seemingly the same as in the real case, namely

$$f'(z) = \lim_{\Delta \to 0} \frac{f(z+\Delta) - f(z)}{\Delta}$$

where Δ can approach zero from *any* (complex) *direction*. And only when all these limits agree, the function is called DIFFERENTIABLE (at z), the value of the resulting derivative equal to this common limit.

Is there a simple way to establish that a given function is differentiable [we don't want to compare infinitely many limits]? The answer is yes, the two real functions u and v must meet the following, so called **Cauchy-Riemann conditions**:

$$\begin{aligned}\frac{\partial u}{\partial x} &\equiv \frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &\equiv -\frac{\partial u}{\partial y}\end{aligned}$$

Proof: $f(z+\Delta) \equiv u(x+\Delta_x, y+\Delta_y) + iv(x+\Delta_x, y+\Delta_y)$ where $\Delta \equiv \Delta_x + i\Delta_y$. This can be expanded (generalized Taylor) as $u(x,y) + \frac{\partial u}{\partial x}\Delta_x + \frac{\partial u}{\partial y}\Delta_y + ... + iv(x,y) + i\frac{\partial v}{\partial x}\Delta_x + i\frac{\partial v}{\partial y}\Delta_y + ... \Rightarrow$

$$\frac{f(z+\Delta)-f(z)}{\Delta} \approx \frac{\frac{\partial u}{\partial x}\Delta_x + \frac{\partial u}{\partial y}\Delta_y + i\frac{\partial v}{\partial x}\Delta_x + i\frac{\partial v}{\partial y}\Delta_y}{\Delta_x + i\Delta_y}$$

Furthermore [we know from real analysis], *all* limits will agree when the 'horizontal' limit $\lim\limits_{\substack{\Delta_x\to 0\\ \Delta_y=0}}$ and the 'vertical' limit $\lim\limits_{\substack{\Delta_y\to 0\\ \Delta_x=0}}$ do. This implies that $\lim\limits_{\Delta_x\to 0}\frac{\frac{\partial u}{\partial x}\Delta_x + i\frac{\partial v}{\partial x}\Delta_x}{\Delta_x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ must equal $\lim\limits_{\Delta_y\to 0}\frac{\frac{\partial u}{\partial y}\Delta_y + i\frac{\partial v}{\partial y}\Delta_y}{i\Delta_y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$, from which the Cauchy-Riemann conditions easily follow. □

Note that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \equiv \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

when the function is differentiable.

EXAMPLE:

- $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$. Find $f'(z)$ [first check whether it exists].

  Solution (checking C-R): $\frac{\partial}{\partial x}(x^2-y^2) \equiv \frac{\partial}{\partial y}(2xy)\ \surd$ and $\frac{\partial}{\partial x}(2xy) \equiv -\frac{\partial}{\partial y}(x^2 - y^2)\ \surd$.

  Answer: $f'(z)$ is then equal to $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2iy \equiv 2z$. Note that we are getting the same answer as if z were *real*. ■

Along these lines, one can show that in general all **polynomial** functions are *differentiable*, and that the corresponding derivative can be obtained by applying the usual $(z^n)' = nz^{n-1}$ rule.

This follows from the fact that, when f_1 and f_2 are differentiable, so is $f_1 + f_2$ [quite trivial to prove] *and* $f_1 \cdot f_2 \equiv (u_1u_2 - v_1v_2) + i(u_1v_2 + u_2v_1)$.

Proof (of the latter): $\frac{\partial u_1}{\partial x}u_2 + u_1\frac{\partial u_2}{\partial x} - \frac{\partial v_1}{\partial x}v_2 - v_1\frac{\partial v_2}{\partial x} \equiv \frac{\partial u_1}{\partial y}v_2 + u_1\frac{\partial v_2}{\partial y} + \frac{\partial v_1}{\partial y}u_2 + v_1\frac{\partial u_2}{\partial y}$ and $\frac{\partial u_1}{\partial y}u_2 + u_1\frac{\partial u_2}{\partial y} - \frac{\partial v_1}{\partial y}v_2 - v_1\frac{\partial v_2}{\partial y} = \frac{\partial u_1}{\partial x}v_2 + u_1\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial x}u_2 + v_1\frac{\partial u_2}{\partial x}$ ($\Rightarrow$ the **product rule** still applies). □

Similarly, if f is differentiable, so is $\frac{1}{f} \equiv \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$.

Proof: $\frac{\partial u}{\partial x}(u^2+v^2) - 2u(\frac{\partial u}{\partial x}u + \frac{\partial v}{\partial x}v) = -\frac{\partial v}{\partial y}(u^2+v^2) + 2v(\frac{\partial u}{\partial y}u + \frac{\partial v}{\partial y}v)$ and $\frac{\partial u}{\partial y}(u^2+v^2) - 2u(\frac{\partial u}{\partial y}u + \frac{\partial v}{\partial y}v) = \frac{\partial v}{\partial x}(u^2+v^2) - 2v(\frac{\partial u}{\partial x}u + \frac{\partial v}{\partial x}v)$ [each divided by $(u^2+v^2)^2$] ($\Rightarrow$ the **quotient rule** still applies). □

And finally a **composition** [$f_1(f_2(z)) \equiv u_1(u_2,v_2) + iv_1(u_2,v_2)$, sometimes denoted $f_1 \circ f_2$] of two differentiable functions is also differentiable.

Proof: $\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x} = \frac{\partial v_1}{\partial x}\frac{\partial u_2}{\partial y} + \frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial y}$ and $\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial y} = \frac{\partial v_1}{\partial x}\frac{\partial u_2}{\partial x} + \frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial x}$ where both u_1 and v_1 have (u_2,v_2) as arguments ($\Rightarrow$ the **chain rule** still applies). □

▷ In **summary**:

All rational expressions (in z) are differentiable (*everywhere*, except when dividing by zero – the so called SINGULARITIES), and the old differentiation formulas still apply (after the $x \to z$ replacement). This provides us with a huge collection of differentiable functions (later to be extended further). □

The natural question to ask now is: Are there any complex functions which are **not differentiable**? The answer is yes, aplenty as well.

EXAMPLE:

- $f(z) = \bar{z} \equiv x - iy$. The first C-R condition requires $\frac{\partial x}{\partial x} \equiv \frac{\partial(-y)}{\partial y}$ which is obviously not met. This function is *nowhere* differentiable. ■

Similarly one can verify that $|z|$, $Re(z)$ and $Im(z)$ are **nowhere differentiable** (since they have zero $v(x,y)$ component). Thus, any expression involving any of these function is nowhere differentiable in consequence.

If a function is differentiable at a point, *and* also at all points of its (open) neighborhood, the function is called ANALYTIC at that point [$\Rightarrow$ a function can be differentiable at a single, isolated point, but it can be analytic only in some open region]. This subtle distinction is not going to have much impact on us, we will simply take 'analytic' as another name for 'differentiable'.

One **interesting implication** of C-R is that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ (proof is trivial). Functions which meet the corresponding *partial* differential equation are called HARMONIC. This means that both $u(x,y)$ and $v(x,y)$ of an *analytic* function are harmonic, and reverse: given a harmonic function, we can make it a $u(x,y)$ [or $v(x,y)$] of an analytic function by *deriving* the corresponding $v(x,y)$ [or $u(x,y)$].

EXAMPLE:

Given $u(x,y) = e^x\cos(y)$, find the corresponding $f(z)$.

Solution: First we can easily check that the function is harmonic. Then we compute $\frac{\partial v}{\partial x} \equiv -\frac{\partial u}{\partial y} = e^x \sin y$ and $\frac{\partial v}{\partial y} \equiv \frac{\partial u}{\partial x} = e^x \cos y$, from which $v(x,y)$ follows [by the procedure of solving *exact* differential equation]: $v(x,y) = e^x \sin y + c_1(x) = e^x \sin y + c_2(y)$. Making these 'compatible' we get $v(x,y) = e^x \sin y$. Thus $f(z) = e^x(\cos y + i \sin y) \equiv e^x \cdot e^{iy} = e^z$.

We have thus proved that the **exponential function** $e^z \equiv \exp(z)$ is also analytic, with a derivative given by $(e^z)' = e^z$ [which follows easily from our last example].

▶Roots of z ◀

The function $f(z) = \sqrt[n]{z} \equiv z^{\frac{1}{n}}$, where n is an integer, can in general have n possible (distinct) values, all given by

$$\sqrt[n]{r}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$$

(depending on the choice of θ). We normally select its PRINCIPAL value, to make the answer unique.

This (principal-value) function is **analytic** everywhere *except* 0 and the negative x-axis (due to the discontinuity of θ when crossing $-x$). The old

$$(z^{\frac{1}{n}})' = \frac{1}{n} z^{\frac{1}{n}-1}$$

formula still applies, in the analytic region.

We now return to our

▶Exponential Function $f(z) = e^z$ ◀

It is *periodic* in the following sense: $f(z \pm 2k\pi i) \equiv f(z)$. The complex strip $-\pi < y \leq \pi$ is called its FUNDAMENTAL REGION (everywhere else, the values of $e^z$are just repeated). Note that now the function can have negative values, e.g. $e^{i\pi} = \cos\pi + i\sin\pi = -1$. Its derivative equals to e^z, as already mentioned.

Proof: In the previous example we saw that e^z was analytic everywhere. Its derivative when z is real is e^x. The only way to extend this to an analytic expression is to add $i\,y$ to x (making it z), as any other combination of x and y would not be analytic. (This argument applies to any analytic function, which means that we can always use the old formulas for differentiation, just replacing x by z). □

The corresponding **inverse function**

(to e^z) is $w(z) \equiv \ln z$, a solution to $z = e^w$ or, more explicitly, to

$$z = e^{u+iv} = e^u(\cos v + i\sin v)$$

where $w \equiv u + iv$. To solve this equation for u and v, we must express z in its polar form, thus: $r(\cos\theta + i\sin\theta) = e^u(\cos v + i\sin v) \Rightarrow$

$$u = \ln r$$

(this is the usual, *real* logarithm) and

$$v = \theta$$

$\ln z$ is thus a multivalued function of z; to fix that, we define its PRINCIPAL VALUE by taking $-\pi < \theta \leq \pi$ [in which case we call the function $Ln(z)$]. It is analytic everywhere *except* at 0 and negative real values [its derivative is given by the old $\frac{dLn(z)}{dz} = \frac{1}{z}$]. Since, in this manner, one can take a logarithm of *any* complex number, we can now find $Ln(-1) = Ln(1 \cdot e^{i\pi})$ [express the number in its polar form] $= 0 + i\pi$ (purely imaginary).

Using the Ln function, we can now define

▶General Exponentiation◀

$$f(z) \equiv z^a$$

where a is also *complex*. This equals to $(e^{Ln(z)})^a = e^{aLn(z)}$ which is well (and uniquely) defined [in terms of its *principal value* – one could also define the corresponding multivalued function].

This function is thus *analytic* everywhere except the negative x axis and 0 [when a is an integer, we need to exclude only 0 when $a < 0$, and nothing when $a > 0$]. Its **derivative** is of course the old $(z^a)' = az^{a-1}$.

Using this definition we can compute $i^i = e^{i(i\frac{\pi}{2})} = e^{-\frac{\pi}{2}} = 0.20788$ [real!].

Similarly, one can also define the usual ▶Trigonometric Functions◀

$$\sin z \equiv \frac{1}{2i}(e^{iz} - e^{-iz})$$

and

$$\cos z \equiv \frac{1}{2}(e^{iz} + e^{-iz})$$

and the corresponding inverse functions $\arcsin(z)$ and $\arccos(z)$. Since these are of lesser importance to us, we are skipping the respective sections of your textbook.

Chapter summary

Complex differentiation is *trivial* for expressions contains z *only*; we differentiate them as if z were real.

This similarity of real and complex differentiation is a nontrivial (certainly not an automatic) consequence of the algebra of complex numbers; it is also the main reason why extending Calculus to complex numbers is so fruitful (as we will see in the next chapter).

As soon as we find $\bar{z}$, $|z|$, $Re(z)$ or $Im(z)$ in a definition of a complex function, the function is nowhere differentiable.

Chapter 10 COMPLEX FUNCTIONS – INTEGRATION

Similar to differentiation, complex integration of analytic functions will be shown to follow the formulas of *real* integration. But there is an extra bit of good news: the so called CONTOUR INTEGRATION will make complex integration even easier, so that many real integrals can be simplified by going complex.

▶Definition◀

of a **complex integral** is similar to that of a line integral in a plane, with a small but essential modification: instead of taking the dot product of $\mathbf{f}$ and $d\mathbf{r}$, the complex function $f(z)$ and the(two-component) infinitesimal element dz are multiplied using complex algebra, resulting in a complex (i.e. two-component) answer, thus:

$$\int_C f(z)dz \equiv \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

where C is come complex curve.

Note that this definition does *not* require $f(z)$ to be analytic; we can thus integrate *all* (not just analytic) complex function.

The actual integration can be carried out by *parametrizing* C and performing the implied single-variable (dt) integration (we need two of them now).

EXAMPLES:

Evaluate:

1. $\int\limits_C \frac{dz}{z}$, where C is the unit circle centered at 0, traversed counterclockwise.

   Solution: Parametrize z by $z = e^{it}$ where $0 \le t < 2\pi$ [this form is more convenient than the more explicit but equivalent $z = \cos t + \ i\sin t$]. Since $\frac{dz}{dt} = ie^{it}$, we can replace dz by $ie^{it}dt$ [and $\frac{1}{z}$ by e^{-it}].

   Answer: $\int\limits_0^{2\pi} e^{-it} \cdot ie^{it}dt = i\int\limits_0^{2\pi} dt = 2\pi i.$

   [Using the more explicit form of z, we would have to struggle with $\int\limits_0^{2\pi} \frac{-\sin t + i\cos t}{\cos t + i\sin t}dt$ to get the same answer].

2. $\int\limits_C Re(z)dz$, where C is the straight-line segment from 0 to $1+i$.

   Solution: $z = t + it$ with $t \in (0,1) \Rightarrow Re(z) = t$ and $dz = (1+i)dt$.

   Answer: $\int\limits_0^1 t(1+i)dt = (1+i)\left[\frac{t^2}{2}\right]_{t=0}^1 = \frac{1}{2} + \frac{i}{2}.$

3. $\int\limits_C (z-z_0)^m dz$, where m is an integer (of either sign), z_0 is a complex constant, and C is a counterclockwise circle of radius $\rho > 0$ centered at z_0 (this is an extension of Example 1).

   Solution: $z = z_0 + \rho e^{it}$ with $t \in (0, 2\pi) \Rightarrow \int\limits_0^{2\pi} \rho^m e^{imt} \cdot \rho e^{it} dt = \rho^{m+1} \int\limits_0^{2\pi} e^{i(m+1)\,t} dt = \rho^{m+1} \int\limits_0^{2\pi} (\cos[(m+1)\,t] + i \sin[(m+1)\,t])\, dt = 0$, with one important exception: when $m = -1$, we get $\rho^0 \int\limits_0^{2\pi} (\cos 0 + i \sin 0)\, dt = 2\pi i$ [remember this result, it is of special importance].

Integrating **analytic** functions

An *analytic* function $f(z)$ can be integrated by first finding its **anti-derivative** (as if z were *real*, i.e. all of the old formulas and techniques still apply), then evaluating it at the first and the last point of C (each being a complex number), and finally subtracting the former from the latter (the same old procedure).

Proof: $\int\limits_C (u+iv)(dx+idy) = \int\limits_C (udx - vdy) + i \int\limits_C (vdx + udy)$. The last two integrals are, effectively, line integrals in a plane (with $\mathbf{r} \equiv [u, -v]$ and $\mathbf{r} \equiv [v, u]$, respectively). They are both path independent iff the C-R conditions are met. Furthermore, when integrated via their 'potentials', they yield

$$g(x,y)\big|_{(x_0,y_0)}^{(x_1,y_1)} + i\ h(x,y)\big|_{(x_0,y_0)}^{(x_1,y_1)}$$

where g and h clearly meet the C-R conditions as well [we use 0 and 1 as indices of the first and last point of C, respectively]. We can thus write the result as $F(z)|_{z_0}^{z_1}$ where F is analytic, and must agree with the usual antiderivative when z is real. The only way to extend a real function $F(x)$ to an analytic function is by $x \to z$. □

EXAMPLE: Find $\int\limits_C z^2\, dz$, where C is a straight line segment from 0 to $2+i$.

Solution: $= \frac{z^3}{3}\Big|_{z=0}^{2+i} = \frac{(2+i)^3}{3} = \frac{8+3\times 4i+3\times 2i^2+i^3}{3} = \frac{2}{3} + \frac{11}{3} i$ [note that the result is path independent]. ■

Integration is thus very simple for fully analytic functions (i.e. analytic *everywhere*, they are also called ENTIRE functions). Things get a bit more interesting when the function has one or more

▶Singularities◀

EXAMPLE: Integrating $\int\limits_C \frac{dz}{z}$, where C is a curve starting at $-1-i$ and ending at $1+i$ yields *two different results* [$-\pi i$ and πi, respectively] depending on whether we pass to the left or to the right of 0 [we have to *avoid* 0, since the function is singular there].

Thus, we have to conclude that the anti-derivative technique $\int_C \frac{dz}{z} = Ln(z)|_{z=-1-i}^{1+i}$ returns the correct answer only when C does not cross the $-x$ axis (the so called CUT), since $Ln(z)$ is *not* analytic there.

To correctly evaluate the other integral (when C passes to the left of 0), we must define our own $\ln(z)$ function whose cut follows $+x$ (rather than $-x$). This simply means selecting θ from the $[0, 2\pi)$ interval, rather than the 'principal' $(-\pi, \pi]$. Using this function, $\ln(z)|_{z=-1-i}^{1+i}$ returns the correct answer (for all paths to the left of 0). ■

In general: $\int_C f(z)\, dz$ is **path-independent** as long as C is modified *without crossing* any *singularity* of $f(z)$.

This implies:

When C is *closed*

$$\oint_C f(z)\, dz \equiv 0$$

provided that there are *no singularities* of $f(z)$ *inside* C.

$\oint$ is a **notation** we use when integrating over a closed curve, this is also called CONTOUR INTEGRATION.

Now, the **main question** is: How do we evaluate $\oint_C f(z)\, dz$ with *some singularities inside* a closed curve C?

This leads to a very important technique of the so called

Contour integration

We are in a good position to figure out the answer:

Each such 'contour' (C) can be subdivided into as many pieces (also closed curves) as there are singularities (with one singularity in each).

Each of these pieces can be then continuously modified (*without crossing* any *singularity)* until its singularity is encircled. This will not change the value of the original integral, which has now become a sum of several 'circle' integrals.

And the value of each of these can be computed by expanding $f(z)$ at z_0 [the corresponding singular point], thus:

$$f(z) = ... + \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ...$$

[the so called LAURENT SERIES, where a_i are constant coefficients], and then integrating term by term.

We already know that only the $\frac{a_{-1}}{z-z_0}$ term will contribute a nonzero value of $2\pi i a_{-1}$. The a_{-1} coefficient of the Laurent expansion is thus of a rather special importance. It is called the **residue** of $f(z)$ at z_0, and is usually denoted by $\underset{z=z_0}{Res}(f)$.

▷ The **final result**: $\oint_C f(z)\, dz$ is equal to $2\pi i$, multiplied by the *sum of residues* of all singular points of $f(z)$ *inside* C [no actual integration is thus necessary].

►Computing Residues◄

▷ A **special** (but very common and important) **case** is

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where $g(z)$ is *analytic* at z_0 [i.e. the singularity at z_0 is due to an explicit division by $(z - z_0)^m$]. We can thus expand $g(z)$ at z_0 in a regular (Taylor) manner, divide the result by $(z - z_0)^m$, and clearly see that

$$\underset{z=z_0}{Res}(f) = \frac{g^{(m-1)}(z_0)}{(m-1)!}$$

EXAMPLES:

1. The residue of $\frac{e^z}{z^2}$ at $z = 0$ is $(e^z)'|_{z=0} = 1$. We can thus easily evaluate $\oint\limits_C \frac{e^z}{z^2}\,dz$ [where C is any contour encircling 0 counterclockwise] as $2\pi i$.

2. Find $\oint\limits_C \frac{1+z^2}{z^2-1}\,dz$, where C is (counterclockwise):

   (a) The unit circle centered at 1.
   Solution: The only singularities of $\frac{1+z^2}{(z+1)(z-1)}$ are at $z = -1$ (residue equal to $\frac{1+(-1)^2}{-1-1} = -1$) and at $z = 1$ (residue equal to $\frac{1+1^2}{1+1} = 1$).
   Answer: $2\pi i$.

   (b) The unit circle centered at -1.
   Answer: $2\pi i \times (-1) = -2\pi i$.

   (c) The unit circle centered at i.
   Answer: 0 (no singularity is inside this C).

   (d) The circle of radius 3, centered at 0.
   Answer: $2\pi i \times (1 - 1) = 0$ [both singularities are inside this C].

3. Identify the singularities (and find the corresponding residues) of $\frac{z^2-1}{z^2+1}$.

   Solution: $\frac{z^2-1}{(z+i)(z-i)}$ has singularities at $z = -i$ (residue: $\frac{(-i)^2-1}{-i-i} = -i$) and at $z = i$ (residue: $\frac{i^2-1}{i+i} = i$).

4. Same for $\frac{z^2+1}{(4z-1)^2}$.

   Solution: $\left[\frac{z^2+1}{16}\right]'_{z=\frac{1}{4}} = \frac{1}{32}$.

5. Same for $\frac{(z+4)^3}{z^4+5z^3+6z^2}$.

   Solution: $\frac{(z+4)^3}{z^2(z+2)(z+3)}$ has singularities at $z = -2$ (residue: $\frac{(-2+4)^3}{(-2)^2(-2+3)} = 2$), at $z = -3$ (residue: $\frac{(-3+4)^3}{(-3)^2(-3+2)} = -\frac{1}{9}$) and at $z = 0$ (residue: $\left[\frac{(z+4)^3}{z^2+5z+6}\right]'_{z=0} = \left.\frac{3(z+4)^2(z^2+5z+6)-(2z+5)(z+4)^3}{(z^2+5z+6)^2}\right|_{z=0} = \frac{3\times 4^2\times 6-5\times 4^3}{6^2} = -\frac{8}{9}$). ■

$\triangleright$ The **general case** of

$$f(z) = \frac{g(z)}{h(z)}$$

where $h(z_0) = 0$. The corresponding residue (at $z = z_0$) equals

$$\lim_{z \to z_0} \left[\frac{f(z)(z - z_0)^m}{(m-1)!} \right]^{(m-1)}$$

where m is the order of the z_0 root. If this can not be established in advance, one has to try $m = 1$, $m = 2$, $m = 3$, ..., until the limit is *finite* (a larger value of m would still yield the correct answer, but with a lot more effort).

EXAMPLES:

Find the residue of

1. $\frac{e^{z^2}}{\cos \pi z}$ at $z = \frac{1}{2}$.

   Solution: $\lim\limits_{z \to \frac{1}{2}} (z - \frac{1}{2}) \cdot \frac{e^{z^2}}{\cos \pi z}$ equals, by L'Hopital rule, $\left. \frac{e^{z^2} + (z - \frac{1}{2}) 2z e^{z^2}}{-\pi \sin \pi z} \right|_{z = \frac{1}{2}} = -\frac{e^{\frac{1}{4}}}{\pi} = -0.40872$.

2. $\dfrac{2z+1}{(1 - e^z)^2}$ at $z = 0$.

   Solution: This looks like a second-order singularity, so we first find $\left(\frac{z^2(2z+1)}{(1-e^z)^2} \right)' = 2 \left(\frac{z}{1-e^z} \right)^2 + 2 \cdot \frac{z}{1-e^z} \cdot \frac{1-e^z+ze^z}{(1-e^z)^2} \cdot (1 + 2z)$ and then take the $z \to 0$ limit. Using L'Hopital rule gives (individually) $\frac{z}{1-e^z} \overset{\prime}{\longrightarrow} \frac{1}{-e^z} \overset{z=0}{\longrightarrow} -1$ and $\frac{1-e^z+ze^z}{(1-e^z)^2} \overset{\prime}{\longrightarrow} \frac{ze^z}{-2e^z(1-e^z)} \overset{\prime}{\longrightarrow} \frac{e^z+ze^z}{-2(e^z-2e^{2z})} \overset{z=0}{\longrightarrow} \frac{1}{2}$.

   Answer: $2(-1)^2 + 2 \times (-1) \times \frac{1}{2} \times 1 = 1$.

   An alternate, and in many cases easier, approach is to directly expand: $\frac{1+2z}{(-z-\frac{z^2}{2}-\frac{z^3}{6}-...)^2} = (1+2z) \frac{1}{z^2} (1 + \frac{z}{2} + \frac{z^2}{6} + ...)^{-2} = (1+2z) \frac{1}{z^2} (1 - z - \frac{z^2}{3} + 3\frac{z^2}{4} + ...) = \frac{1}{z^2} + \frac{1}{z} - \frac{19}{12} + ...$.

3. $\frac{1}{1 - \cos z}$ at $z = 0$.

   Solution: Since $\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - ...$ we can deduce that the singularity is of second order. The residue is thus $\left(\frac{z^2}{1 - \cos z} \right)' = \frac{2z(1-\cos z) - z^2 \sin z}{(1-\cos z)^2}$, after we take the $z \to 0$ limit. This requires applying L'Hopital rule *four* times (relatively easy if we differentiate and substitute at the same time), resulting in 0 for the numerator, and $2 \times \binom{3}{1} \times \cos 0 \times \cos 0 = 6$ for the denominator (we need to know that it is non-zero).

   Answer: 0.

   Alternately, we expand $(\frac{z^2}{2} - \frac{z^4}{24} - ...)^{-1} = \frac{2}{z^2} (1 - \frac{z^2}{12} + ...)^{-1} = \frac{2}{z^2} (1 + \frac{z^2}{12} + ...) = \frac{2}{z^2} + \frac{1}{6} + ...$, yielding the same result.

4. $\dfrac{1}{1+z^3}$ at $z_s = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Solution: Even though this is a rational function, we may now prefer to do $\lim\limits_{z=z_s} \left(\frac{z-z_s}{1+z^3}\right) =$ [L'Hopital] $\frac{1}{3z^2}\Big|_{z=z_s} = -\frac{z_s}{3}$ [since $z_s^3 = -1$] $= -\frac{1}{6} - i\frac{\sqrt{3}}{6}$.

5. $\frac{1}{e^z-1-z}$ at $z = 0$.

Solution: Here we would rather expand: $\frac{1}{\frac{z^2}{2}+\frac{z^3}{6}+...} = \frac{2}{z^2}\cdot(1+\frac{z}{3}+...)^{-1} = \frac{2}{z^2}\cdot(1-\frac{z}{3}+...) \Rightarrow$

Answer: $-\frac{2}{3}$ [coefficient of $\frac{1}{z}$]. ■

Applications

to evaluating *real* integrals of several special types:

▶Rational Functions of $\sin t$ and/or $\cos t$ ◀

integrated over a *full-period* interval, i.e. from c to $c + 2\pi$ where c is any real number (usually equal to 0).

We introduce $z = e^{it}$ ($\Rightarrow \sin t = \frac{z-z^{-1}}{2i}$, $\cos t = \frac{z+z^{-1}}{2}$ and $dt = \frac{dz}{iz}$) and integrate the corresponding complex function over C_0 [the counterclockwise unit circle centered at 0] via contour integration (i.e. by finding all singularities inside this circle and adding their residues $\times 2\pi i$; note that the final answer must be *real*).

EXAMPLES:

1. $\int\limits_0^{2\pi} \frac{dt}{5-3\cos t} = \oint\limits_{C_0} \dfrac{dz}{iz(5-3\frac{z+z^{-1}}{2})} = \oint\limits_{C_0} \dfrac{dz}{i(5z-\frac{3}{2}z^2-\frac{3}{2})} = -\frac{2}{3i}\oint\limits_{C_0} \dfrac{dz}{z^2-\frac{10}{3}+1} = -\frac{2}{3i}\oint\limits_{C_0} \dfrac{dz}{(z-3)(z-\frac{1}{3})} = -\frac{2}{3i} \times \frac{1}{\frac{1}{3}-3} \times 2\pi i = \frac{\pi}{2}$ [the only singularity inside C_0 is at $z = \frac{1}{3}$, $\frac{1}{\frac{1}{3}-3}$ being the corresponding residue].

Note that $\int\limits_0^{4\pi} \frac{dt}{5-3\cos t}$ would be simply twice as large. Similarly, $\int\limits_0^{\pi} \frac{dt}{5-3\cos t}$ would yield *half* the value, because $\frac{1}{5-3\cos t}$ is an *even* function of t.

2. $\int\limits_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}\,d\theta$ [θ plays the rôle of t] $= \oint\limits_{C_0} \dfrac{1+\frac{z-z^{-1}}{2i}}{3+\frac{z+z^{-1}}{2}}\cdot\dfrac{dz}{iz} = -\oint\limits_{C_0} \dfrac{z^2+2iz-1}{z(z^2+6z+1)}\,dz = -\oint\limits_{C_0} \dfrac{z^2+2iz-1}{z(z+3+\sqrt{8})(z+3-\sqrt{8})}\,dz.$

The singularities are at $z = 0$ [residue: -1] and $z = -3+\sqrt{8}$ [residue: $\frac{(-3+\sqrt{8})^2+2i(-3+\sqrt{8})-1}{(-3+\sqrt{8})\times 2\sqrt{8}} = 1 + \frac{i}{\sqrt{8}}$].

Answer: $-\frac{i}{\sqrt{8}} \times 2\pi i = \frac{\pi}{\sqrt{2}}$.

3. $\int_0^{\pi} \frac{\sin^2 t\ - 2\cos t}{2+\cos t}\, dt = \frac{1}{2}\int_0^{2\pi} \frac{\sin^2 t\ - 2\cos t}{2+\cos t}\, dt$ [even function] $=$
$\frac{1}{2}\oint_{C_0} \frac{\left(\frac{z-z^{-1}}{2i}\right)^2 - 2\frac{z+z^{-1}}{2}}{2+\frac{z+z^{-1}}{2}} \cdot \frac{dz}{iz} = -\frac{1}{4i}\oint_{C_0} \frac{z^4+4z^3-2z^2+4z+1}{z^2(z^2+4z+1)}\, dz.$
The singularities are at $z = 0$ (residue: $\left[\frac{z^4+4z^3-2z^2+4z+1}{z^2+4z+1}\right]'_{z=0} =$ $\frac{4\times 1 - 1\times 4}{1^2} = 0$) and at $z = -2+\sqrt{3}$ (residue: $\frac{z_s^4+4z_s^3-2z_s^2+4z_s+1}{z_s^2(z_s+2+\sqrt{3})} =$ $\frac{-z_s^2-2z_s^2-z_s^2}{z_s^2\times 2\sqrt{3}} = \frac{-2}{\sqrt{3}}$, where $z_s \equiv -2+\sqrt{3}$; note that $z_s^2+4z_s \equiv -1$ and $4z_s+1 \equiv -z_s^2$).

Answer: $-\frac{1}{4i} \times \frac{-2}{\sqrt{3}} \times 2\pi i = \frac{\pi}{\sqrt{3}}$.

4. $\int_0^{\pi} \frac{dx}{a+\cos^2 x}$ where $a > 0$.

Solution: $= \frac{1}{2}\int_{-\pi}^{\pi} \frac{dx}{a+\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2} = \frac{1}{2}\oint_{C_0} \frac{\frac{dz}{iz}}{a+(\frac{z+z^{-1}}{2})^2} = \frac{2}{i}\oint_{C_0} \frac{z\, dz}{4az^2+(1+z^2)^2}$.

The singularities are the roots of $z^4+(2+4a)z^2+1 = 0$ namely: $z^2 = -(1+2a)\pm 2\sqrt{a+a^2}$ [the minus sign puts us outside the unit circle, the plus sign results in a negative value between 0 and -1, the proof is simple].

The roots we need are thus $z_{1,2} = \pm i\sqrt{1+2a-2\sqrt{a+a^2}}$, the corresponding factorization of the function's denominator results in
$\frac{z}{(z-z_1)(z-z_2)(z^2+1+2a+2\sqrt{a+a^2})}$.

The residues are thus $\frac{z_1}{(z_1-z_2)\cdot 4\sqrt{a+a^2}}$ and $\frac{z_2}{(z_2-z_1)\cdot 4\sqrt{a+a^2}}$ The sum is $\frac{1}{4\sqrt{a+a^2}}$, multiplied by $2\pi i \times \frac{2}{i}$ equals $\frac{\pi}{\sqrt{a+a^2}}$. ■

▶Rational Function of x ◀

(the denominator must be a polynomial of at least two degrees higher than that in the numerator), integrated from $-\infty$ to ∞.

We replace any such integral by a complex integral over z ($\equiv x$), from $-R$ to R, *extended* by the half circle Re^{it} [$t \in (0,\pi)$] to make the contour closed. One can show that, in the $R \to \infty$ limit, the half circle's contribution tends to 0, and one thus obtains the correct answer to the original integral.

Proof: One can easily show that the magnitude of a complex integral cannot exceed the maximum magnitude of the integrand, multiplied by the length of C. In our case, the length of the half circle is πR, and the function's magnitude has an upper bound (based on the triangular inequality) equal to a polynomial in R over another polynomial of at least two degrees higher. The product of the πR length and this upper bound tends to 0 as $R \to \infty$. □

▷ **Algorithm**: To find the value of the original integral, one thus has to replace x by z in the integrand, find all its *singularities* in the *upper half plane* ($y > 0$), *add* their *residues*, and multiply by $2\pi i$.

Note that the answer must be *real*.

EXAMPLES:

Evaluate:

1. $\int\limits_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

   Solution: $\sqrt[4]{-1} = \frac{\pm 1 \pm i}{\sqrt{2}}$ are the function's singularities, $\frac{\pm 1 + i}{\sqrt{2}}$ being in the upper half plane. The corresponding residues are found from $\lim\limits_{z \to z_s} \left(\frac{z - z_s}{1+z^4}\right) =$ [by L'Hospital] $\frac{1}{4z_s^3} \equiv -\frac{z_s}{4}$ [since $z_s^4 \equiv -1$], where z_s is either one of the two singularities. Substituting $z_s = \frac{1+i}{\sqrt{2}}$ this yields $-\frac{1+i}{4\sqrt{2}}$, substituting $z_s = \frac{-1+i}{\sqrt{2}}$ we get $\frac{1-i}{4\sqrt{2}}$.

   Answer: $\left(-\frac{1+i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}}\right) \times 2\pi i = \frac{\pi}{\sqrt{2}}$.

2. $\int\limits_{-\infty}^{\infty} \frac{dx}{1+x^6}$.

   Solution: The relevant singularities are $\sqrt[6]{-1} = \frac{\pm\sqrt{3}+i}{2}$ and i. The corresponding residues are given by $\lim\limits_{z \to z_s} \left(\frac{z - z_s}{1+z^6}\right) = \frac{1}{6z_s^5} = -\frac{z_s}{6}$ [since $z_s^6 \equiv -1$]. The sum of the three residues is thus $-\frac{i+i}{6} = -\frac{i}{3}$.

   Answer: $-\frac{i}{3} \times 2\pi i = \frac{2}{3}\pi$.

3. $\int\limits_{0}^{\infty} \frac{dx}{(1+x^2)^3}$.

   Solution: This equals to $\frac{1}{2} \int\limits_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$ since the integrand is even. Furthermore, $\frac{1}{(z-i)^3\,(z+i)^3}$ has only one $y > 0$ singularity ($z_s = i$), which is of the *third* order. The corresponding residue is thus $\frac{1}{2}\left(\frac{1}{(z+i)^3}\right)''_{z=i} = 6(z+i)^{-5}\big|_{z=i} = \frac{6}{(2i)^5} = \frac{3}{16i}$.

   Answer: $\frac{3}{16i} \times 2\pi i = \frac{3}{8}\pi$.

4. $\int\limits_{0}^{\infty} \frac{1+x^2}{1+x^4}\, dx$.

   Solution: $= \frac{1}{2} \int\limits_{-\infty}^{\infty} \frac{1+x^2}{1+x^4}\, dx$ [even]. The relevant singularities are at $\frac{\pm 1+i}{\sqrt{2}}$, the corresponding residues are given by $\lim\limits_{z=z_s} \left(\frac{(1+z^2)(z-z_s)}{1+z^4}\right) = \frac{1+z_s^2}{4z_i^3} = -\frac{1+z_s^2}{4} \cdot z_s$ [as $z_s^4 \equiv -1$]. Their sum equals $-\frac{1+\frac{(1+i)^2}{2}}{4} \cdot \frac{1+i}{\sqrt{2}} - \frac{1+\frac{(-1+i)^2}{2}}{4} \cdot \frac{-1+i}{\sqrt{2}} = -\frac{(1+i)^2}{4\sqrt{2}} - \frac{(1-i)(-1+i)}{4\sqrt{2}} = -\frac{2i}{4\sqrt{2}} - \frac{2i}{4\sqrt{2}} = \frac{-i}{\sqrt{2}}$.

Answer: $\frac{1}{2}\cdot\frac{-i}{\sqrt{2}}\cdot 2\pi i=\frac{\pi}{\sqrt{2}}$. ■

We now consider integrating, from $-\infty$ to ∞, the same type of rational expression as in the previous case, further

▶Multiplied by $\sin kx$ or $\cos kx$ ◀

Now we replace $\sin kx$ (or $\cos kx$) by e^{ikz}, find the value of the corresponding integral in the same manner as before, and then take the imaginary (or real) part of the answer.

EXAMPLES:

Find the value of:

1. $\int_{-\infty}^{\infty}\frac{\sin 2x}{x^2+x+1}\,dx$.

   Solution: $= Im\int_{-\infty}^{\infty}\dfrac{e^{2iz}}{z^2+z+1}\,dz$. The last integral can be evaluated by adding the $y>0$ residues of the integrand and multiplying their sum by $2\pi i$. The only contributing singularity is at $z_s=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, the corresponding residue equals $\dfrac{e^{2iz_s}}{(z_s-\bar{z}_s)}=\frac{e^{-\sqrt{3}}(\cos 1-i\sin 1)}{i\sqrt{3}}$. Multiplying this by $2\pi i$ and keeping the imaginary part only results in $-\frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}}\sin 1=-0.54006$

2. Similarly, $\int_{-\infty}^{\infty}\frac{\cos 2x}{(x^2+4)^2}\,dx = Re\int_{-\infty}^{\infty}\dfrac{e^{2iz}}{(z^2+4)^2}\,dz$, with $z_s=2i$ being (the only relevant) second-order singularity, having a residue of $\left[\dfrac{e^{2iz}}{(z+2i)^2}\right]'_{z=2i}=$ $\left.\dfrac{2ie^{2iz}(z+2i)^2-2(z+2i)e^{2iz}}{(z+2i)^4}\right|_{z=2i}=\dfrac{2i(4i)^2-2(4i)}{(4i)^4}\cdot e^{-4}=-\frac{5i}{32}\cdot e^{-4}$. Multiplying this by $2\pi i$ and keeping the real part of the result yields $\frac{5\pi}{16}\cdot e^{-4}=$ 0.017981. ■

▶Other Cases◀

EXAMPLES:

1. $I\equiv\int_{-\infty}^{\infty}\dfrac{e^{mx}}{1+e^x}\,dx$ where $0<m<1$.

   Solution. We make x complex ($x\to z$) and integrate the same function over the following contour [a collection of four straight-line segments which we call C_1, C_2, C_3 and C_4]: $-R$ to R (real), R to $R+2\pi i$, $R+2\pi i$ to $-R+2\pi i$, and $-R+2\pi i$ to $-R$. One can show that, on C_3, the integrand equals $\dfrac{e^{mx+2\pi mi}}{1+e^{x+2\pi i}}\equiv$ $e^{2\pi mi}\cdot\dfrac{e^{mx}}{1+e^x}$. Since $\left|\dfrac{e^{mz}}{1+e^z}\right|\le\dfrac{e^{mR}}{e^R-1}$ on C_2 and $\left|\dfrac{e^{mz}}{1+e^z}\right|\le\dfrac{e^{-mR}}{1-e^{-R}}$ on C_3, their contributions disappear in the $R\to\infty$ limit. In the same limit, the contribution of C_1 yields I, and that of C_3 results in $-e^{2\pi mi} \cdot I$ [since we are going backwards].

The contour integral has only one singularity at $z = \pi i$, with the residue equal to $\lim_{z=\pi i} \left(\frac{(z-\pi i)\, e^{mz}}{1+e^z} \right) = \frac{e^{m\pi i}}{e^{\pi i}} = -e^{\pi mi}$. Its value is thus $-2\pi i e^{\pi mi}$; the value of our I must be this, divided by $1 - e^{2\pi mi}$, i.e. $\frac{-2\pi i e^{\pi mi}}{1-e^{2\pi mi}} \equiv \frac{2\pi i}{e^{\pi mi} - e^{-\pi mi}} = \frac{\pi}{\sin(m\pi)}$.

2. $I \equiv \int\limits_0^\infty \frac{x^{p-1}}{1+x}\, dx$ where $0 < p < 1$. [Note that this integral can be converted to the previous one by a $x = e^u$ substitution, but we will pretend not to notice].

   Solution: This time we use C_1: ri to $R + ri$ (straight line), C_2: $R + ri$ to $R - ri$ (nearly a full circle centered on 0), C_3: $R - ri$ to $-ri$ (straight line), and C_4: $-ri$ to ri (a semicircle centered at 0). Since $\left| \frac{z^{p-1}}{1+z} \right| \leq \frac{R^{p-1}}{R-1}$ on C_2, this contribution disappears in the $R \to \infty$ limit, since $\left| \frac{z^{p-1}}{1+z} \right| \leq \frac{r^{p-1}}{1-r}$ on C_4, ditto for the $r \to 0$ limit.

   Secondly, on C_3, $\frac{z^{p-1}}{1+z} = \frac{e^{(p-1)(\ln x + 2\pi i)}}{1+x} \equiv e^{2\pi pi} \cdot \frac{x^{p-1}}{1+x}$ [in the $r \to 0$ limit], so its contribution is $-e^{2\pi pi} \times I$. The only singularity inside the contour is at $z = -1$, with the residue of $(-1)^{p-1} = e^{i\pi(p-1)}$.

   Answer: $\frac{2\pi i e^{i\pi(p-1)}}{1-e^{2\pi pi}} = \frac{-2\pi i e^{i\pi p}}{1-e^{2\pi pi}} = \frac{\pi}{\sin(p\pi)}$.

3. Compute $\int\limits_0^\infty \cos(x^2)\, dx$ and $\int\limits_0^\infty \sin(x^2)\, dx$ as the real and imaginary part of $\int\limits_0^\infty e^{ix^2}\, dx$.

   Solution: Our segments are now: C_1 [0 to R, a straight line], C_2 [$Re^{i\frac{\pi}{4}t}$ with $0 \leq t \leq 1$, an eighth of a circle], and C_3 [$Re^{i\pi/4}$ to 0, a straight line]. On C_2 $\left| \int\limits_{C_2} e^{iz^2}\, dz \right| \leq \int\limits_{C_2} \left| e^{iz^2} \right| |dz| = \int\limits_0^1 e^{-R^2 \sin\frac{\pi}{2}t} R\frac{\pi}{4} dt \leq \frac{\pi}{4} R \int\limits_0^1 e^{-R^2 t}\, dt = \frac{\pi}{4} R \left. \frac{e^{-R^2 t}}{-R^2} \right|_{t=0}^1 \leq \frac{\pi}{4} \cdot \frac{1}{R} \to 0$ as $R \to \infty$.

   Since our integrand has no singularities, the contributions of C_1 and C_3 must be identical, with opposite signs (to cancel out). Parametrizing C_3 by $z = t(1+i)$ with $0 \leq t \leq \infty$ (taking the $R \to \infty$ limit at the same time) results in $(1+i) \int\limits_0^\infty e^{-2t^2}\, dt$ [as $z^2 = 2it^2$, and $dz = (1+i)\, dt$] $= (1+i) \cdot \sqrt{\frac{\pi}{8}}$. This implies that $\int\limits_0^\infty \cos(x^2)\, dx = \int\limits_0^\infty \sin(x^2)\, dx = \frac{\sqrt{2\pi}}{4}$.