# ORDINARY DIFFERENTIAL EQUATIONS
## First-Order

**'Trivial':**

$$y' = f(x)$$

Solution:

$$y(x) = \int f(x)dx + C$$

**Separable:**

$$y' = h(x) \cdot g(y)$$

Solution:

$$\int \frac{dy}{g(y)} = \int h(x)dx + C$$

**Scale-independent:**

$$y' = g\left(\frac{y}{x}\right)$$

Solution:

$$u = \frac{y}{x}$$

$$\int \frac{du}{g(u) - u} = \int \frac{dx}{x} + C$$

**Modified Scale-Independent**

$$y' = \frac{y}{x} + g\left(\frac{y}{x}\right) \cdot h(x)$$

Solution:

$$u = \frac{y}{x}$$

$$\int \frac{du}{g(u)} = \int h(x)\frac{dx}{x} + C$$

**Other Substitution (suggested)....**
**Linear**

$$y' + g(x) \cdot y = r(x)$$

Variation of Parameters:

$$y(x) = c(x) \cdot e^{-\int g(x)dx}$$

and substitute, getting trivial eq. fro $c(x)$.

**Bernoulli:**

$$y' + f(x) \cdot y = r(x) \cdot y^a$$

Solution:

$$u = y^{1-a}$$

$$u' + (1-a)f(x) \cdot u = (1-a)r(x)$$

(linear).

**Exact:**

$$g(x,y)\,dx + h(x,y)\,dy = 0$$

Test:

$$\frac{\partial g}{\partial y} \equiv \frac{\partial h}{\partial x}$$

If it is, solution constructed by:

$$G(x,y) = \int g(x,y)\,dx$$

$$H(y) = h(x,y) - \frac{\partial G}{\partial y}$$

$$G(x,y) + \int H(y)\,dy = C$$

If it is not

$$P(x,y)dx + Q(x,y)dy = 0$$

find an integrating factor either from

$$\frac{d\ln F}{dx} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}$$

(must be a function of $x$ only - no $x$), or from

$$\frac{d\ln F}{dy} = \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P}$$

(must be a function of $y$ - no $x$).

**Clairaut:**

$$y = xy' + g(y')$$

Solution:

$$y = xC + g(C)$$

(regular), and

$$x = -g'(p)$$

solved for $p$ and substituted back into

$$y = xp + g(p)$$

(singular - envelope of regular).

**Final trick:**

$$\begin{array}{ccc} x & \leftrightarrow & y \\ y' & \rightarrow & \frac{1}{y'} \end{array}$$

solve, then

$$x \leftrightarrow y$$

**Orthogonal families of curves:**

A family is described by an equation involving $y$, $x$ and $C$.

Convert into a differential equation (differentiate the equation with respect to $x$, eliminate $c$) :

$$y' = f(x, y)$$

## Second Order

1. initial conditions:

$$\begin{aligned} y(x_0) &= a \\ y'(x_0) &= b \end{aligned}$$

2. boundary conditions:

$$\begin{aligned} y(x_1) &= a \\ y(x_2) &= b \end{aligned}$$

**Reducible to first order.**
i) missing $y$

$$y' \equiv z$$

ii) missing $x$

$$z \equiv y'$$

and consider $y$ to be the *independent* variable!

$$y'' \to \frac{dz}{dy} \cdot z$$

**LINEAR**

$$y'' + f(x)y' + g(x)y = r(x)$$

1. General solution:

$$y = C_1 y_1 + C_2 y_2 + y_p$$

2. When $y_1$ is known, $y_2$ **and** $y_p$ can be found by variation of parameters (VP):

$$y = c(x)y_1$$

(substitute in the homogeneous equation), and

$$y_p = u(x)y_1 + v(x)y_2$$

where

$$u' = \frac{y_2 r}{y_2 y_1' - y_1 y_2'}$$

and

$$v' = \frac{y_1 r}{y_1 y_2' - y_2 y_1'}$$

**With constant coefficients**

$$y'' + ay' + by = r(x)$$

Homogeneous Case

$$\lambda^2 + a\lambda + b = 0$$

(characteristic polynomial).

1. Two (distinct) real roots.

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

2. Complex conjugate roots $\lambda_{1,2} = p \pm i\,q$.

$$\begin{aligned} y_1 &= e^{px}\cos(qx) \\ y_2 &= e^{px}\sin(qx) \end{aligned}$$

3. Double root.

$$y_1 = e^{-\frac{a}{2}x}$$

$$\begin{aligned} y_1 &= e^{\lambda x} \\ y_2 &= xe^{\lambda x} \end{aligned}$$

Non-homogeneous Case
Use the V of P formulas.

**Special cases** of $r(x)$

- polynomial

$y_p$ is a polynomial of the *same degree* but with *undetermined coefficients*.
When $\lambda = 0$, $y_p$ must be further multiplied by $x^\ell$, where $\ell$ is the multiplicity of 0.

- exponential, or

$$ke^{\alpha x}$$

$$y_p = Ae^{\alpha x}$$

When $\alpha = \lambda$ [any of the roots], $y_p$ must be multiplied by $x^\ell$ ($\ell$ is the multiplicity of $\alpha$).

- trigonometric case:

$$k_s e^{px}\sin(qx) + k_c e^{px}\cos(qx)$$

$$y_p = [A\sin(qx) + B\cos(qx)]e^{px}$$

When $\lambda = p \pm i\,q$, the trial solution must be multiplied by $x$.

- general case

$$P_n(x)e^{\alpha x}$$

$$y_p = Q_n(x)e^{\alpha x}$$

  When $\alpha$ coincides with a root (of multiplicity $\ell$) this must be further multiplied by $x^\ell$.

- Superposition principle

**CAUCHY**

$$x^2y'' + axy' + by = r(x)$$

- Introduce a new *independent* variable

$$t = \ln x$$

  We have already derived the following set of formulas for performing such a conversion:

$$y' \to \frac{\dot{y}}{x}$$

  and

$$y'' \to \frac{\ddot{y}}{x^2} - \frac{\dot{y}}{x^2}$$

  resulting in

$$\ddot{y} + (a-1)\dot{y} + by = r(e^t)$$

  (linear).

- Direct technique:

more convenient when $r = 0$, or does *NOT* have the *special* form.
Solve

$$m^2 + (a-1)m + b = 0$$

Homogeneous solution:

$$y = Ax^{m_1} + Bx^{m_2}$$

**Third and Higher-Order Linear ODEs**

$$y''' + f(x)y'' + g(x)y' + h(x)y = r(x)$$

Solution:

$$y = C_1y_1 + C_2y_2 + C_3y_3 + y_p$$

Vof P for constructing $y_p$

$$u' = \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ r & y_2'' & y_3'' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}$$

**Constant coefficients**

Simple extension of second-order equations.

**Sets of Linear, First-Order, Constant-Coefficient ODEs**

**Matrices**

Be able to convert a matrix to *our* matrix echelon form (use 'gausselim' first) and thus solve a linear set of $n$ equations for $m$ unknowns.

**Eigenvalues & Eigenvectors**

Find all roots of the following characteristic polynomial

$$\begin{aligned}
&\det(\lambda\mathbb{I} - \mathbb{A}) = \lambda^n - \lambda^{n-1} \cdot \mathrm{Tr}(\mathbb{A}) \\
&+\lambda^{n-2} \cdot \{\text{sum of all } 2 \times 2 \text{ major subdeterminants}\} \\
&- \lambda^{n-3} \cdot \{\text{sum of all } 3 \times 3 \text{ major subdeterminants}\} \\
&+ .... \pm \det(\mathbb{A})
\end{aligned}$$

For each root (eigenvalue), find all linearly independent eigenvectors which meet

$$(\mathbb{A} - \lambda\mathbb{I})\mathbf{q} = \mathbf{0}$$

There must be at least one, but not more than the multiplicity of the eigenvalue.

**Solving a homogenous set of ODEs**

$$\mathbf{y}' = \mathbb{A}\mathbf{y}$$

(know how to deal with $\mathbb{B}\mathbf{y}' = \mathbb{A}\mathbf{y}$).

Basic solutions are based on eigenvalue - eigenvector pairs:

$$\mathbf{q} \cdot e^{\lambda x}$$

In most cases, this will do the job.

The only difficulty arrises when a multiple eigenvalue has fewer than $\ell$ (the multiplicity) eigenvectors.

The 'missing' basic solutions are constructed by

$$(\mathbf{q}x + \mathbf{s})e^{\lambda x}$$

where $\mathbf{s}$ is found from

$$(\mathbb{A} - \lambda\mathbb{I})^2 \mathbf{s} = \mathbf{0}$$

(this will have *more* linearly independent solutions than $(\mathbb{A} - \lambda\mathbb{I})\mathbf{q} = \mathbf{0}$ did - this tells us how many new basic solutions we have), and

$$\mathbf{q} = (\mathbb{A} - \lambda\mathbb{I})\mathbf{s}$$

(must be non-zero)!

And, if still not done, we have to proceed to

$$(\mathbf{q}\frac{x^2}{2!} + \mathbf{s}x + \mathbf{u})e^{\lambda x}$$

where $\mathbf{u}$ is a solution to

$$(\mathbb{A} - \lambda\mathbb{I})^3 \mathbf{u} = \mathbf{0}$$

and

$$\begin{aligned} \mathbf{s} &= (\mathbb{A} - \lambda\mathbb{I})\mathbf{u} \\ \mathbf{q} &= (\mathbb{A} - \lambda\mathbb{I})\mathbf{s} \end{aligned}$$

**Complex Case**

The basic solutions will go in complex conjugates, i.e.

$$\begin{aligned} &\mathbf{q} \cdot e^{(a+ib)x} \\ &\mathbf{q}^* \cdot e^{(a-ib)x} \end{aligned}$$

(unacceptable)! They must be replaced by

$$e^{ax}\left[\mathrm{Re}(\mathbf{q})\cos(bx) - \mathrm{Im}(\mathbf{q})\sin(bx)\right]$$
$$e^{ax}\left[\mathrm{Re}(\mathbf{q})\sin(bx) + \mathrm{Im}(\mathbf{q})\cos(bx)\right]$$

**Non-homogeneous case**

$$\mathbf{y}' - \mathbb{A}\mathbf{y} = \mathbf{r}(x)$$

By V of P: Define a matrix

$$\mathbb{Y} \equiv \boxed{\mathbf{y}^{(1)} \,\big|\, \mathbf{y}^{(2)} \,\big|\, \dots \,\big|\, \mathbf{y}^{(n)}}$$

then

$$\mathbf{y}^{(p)} = \mathbb{Y}\int \mathbb{Y}^{-1}\cdot\mathbf{r}(x)\,dx$$

By undetermined coefficients, when

$$\mathbf{r}(x) = \left(\mathbf{a}_k x^k + \mathbf{a}_{k-1}x^{k-1} + \dots + \mathbf{a}_1 x + \mathbf{a}_0\right)e^{\beta\,x}$$

Use the following trial solution:

$$\left(\mathbf{b}_m x^m + \mathbf{b}_{m-1}x^{m-1} + \dots + \mathbf{b}_1 x + \mathbf{b}_0\right)e^{\beta\,x}$$

where $m$ equals $k$ *plus* the number of levels of $\beta$ as an eigenvalue of $\mathbb{A}$.

When $\beta$ does *not* coincide with an *eigenvalue* of $\mathbb{A}$, the equations to solve to obtain $\mathbf{b}_k$, $\mathbf{b}_{k-1}$, ..., $\mathbf{b}_1$ are

$$\begin{aligned}
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_k &= -\mathbf{a}_k \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k-1} &= k\mathbf{b}_k - \mathbf{a}_{k-1} \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k-2} &= (k-1)\mathbf{b}_{k-1} - \mathbf{a}_{k-2} \\
&\vdots \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_0 &= \mathbf{b}_1 - \mathbf{a}_0
\end{aligned}$$

When $\beta$ coincides with a *simple* (as opposed to multiple) eigenvalue of $\mathbb{A}$, we have to solve

$$\begin{aligned}
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k+1} &= \mathbf{0} \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_k &= (k+1)\mathbf{b}_{k+1} - \mathbf{a}_k \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k-1} &= k\mathbf{b}_k - \mathbf{a}_{k-1} \\
&\vdots \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_0 &= \mathbf{b}_1 - \mathbf{a}_0
\end{aligned}$$

where $\mathbf{b}_{k+1}$ is the corresponding eigenvector, multiplied by such a constant as to make the second equation solvable Similarly, when solving the second equation for $\mathbf{b}_k$, a $c$-multiple of the same eigenvector must be added to the solution, to make the third equation is solvable, etc.

The second possibility is

$$\mathbf{r}(x) = \mathbf{P}(x)e^{ax}\cos(bx) + \mathbf{Q}(x)e^{ax}\sin(bx)$$

where $\mathbf{P}(x)$ and $\mathbf{Q}(x)$ are polynomials, and $a + i\,b$ is *not* an eigenvalue of $\mathbb{A}$.

**Power-Series Solution** of

$$y'' + f(x)y' + g(x)y = 0$$

Substitute

$$y = \sum_{i=0}^{\infty} c_i x^i$$

Adjust the indices of all sums to yield the same power of $x$.

Extract and solve the 'exceptional terms.

Set up a recurrence formula for $c_{i+2}$.

Solve it, choosing $c_0 = 1$ and $c_1 = 0$, then $c_0 = 0$ and $c_1 = 1$. Or, if these are given, $c_0 = y(0)$ and $c_1 = y'(0)$.

Identify the two basic solutions, if possible.

Utilize V of P when one solution is simple and the other difficult.

**Method of Frobenius** applies to

$$y'' + \frac{a(x)}{x}y' + \frac{b(x)}{x^2}y = 0$$

Solve the indicial equation

$$r^2 + (a_0 - 1)r + b_0 = 0$$

Substitute

$$\sum_{i=0}^{\infty} c_i x^{r_1+i}$$

using the larger root first. This will *always* yield the first basic solution.

To build the second one, use:

1. For *two distinct* real roots which don't differ by an integer

$$\sum_{i=0}^{\infty} c_i x^{r_2+i}$$

   with $r$ being the smaller root.

2. For a *double* root

$$y_1 \ln x + \sum_{i=0}^{\infty} c_i^* x^{r+i}$$

   ($c_0^*$ can be set to 0).

3. For two roots which *differ* by an *integer*

$$Ky_1 \ln x + \sum_{i=0}^{\infty} c_i^* x^{i+r_2}$$

   (set $c_0^*$ to 1, the coefficient of $x^{r_1}$ will yield $K$, but let you choose $c_{r_1-r_2}^* = 0$).