PREREQUISITES

Polynomials (synthetic division)

$$
\begin{array}{ll}
(x^3 - 3x^2 + 2x - 4) \div (x - 2) = & x^2 - x \ \ [\text{QUOTIENT}] \\
x^3 - 2x^2 & \text{(subtract)} \\
\quad -x^2 + 2x - 4 & \\
\quad -x^2 + 2x & \text{(subtract)} \\
\qquad\qquad -4 & [\text{REMAINDER}]
\end{array}
$$

Rules of exponentiation:

$$
\begin{aligned}
a^A \cdot a^B &= a^{A+B} \\
(a^A)^B &= a^{AB}
\end{aligned}
$$

Note:

$$(a^A)^B \neq a^{(A^B)}$$

Differentiation:

Product rule

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

extended to second derivative

$$(f \cdot g)'' = f'' \cdot g + 2f' \cdot g' + f \cdot g''$$

Integration:

Basic formulas:

$$
\begin{aligned}
\int x^\alpha dx &= \frac{x^{\alpha+1}}{\alpha+1} \qquad \alpha \neq -1 \\
\int e^{\beta x} dx &= \frac{e^{\beta x}}{\beta} \\
\int \frac{dx}{x} &= \ln|x|
\end{aligned}
$$

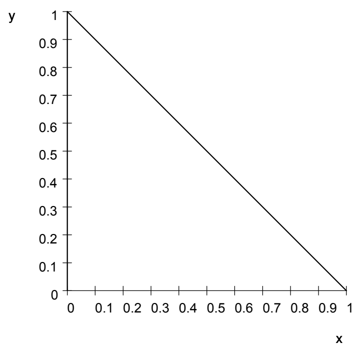
Taylor (Maclaurin) Expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + ...$$

e.g.

$$e^x = 1 + x + x^2/2 + x^3/3! + x^4/4! + ...$$

2-D region (subsequent integration):



Matrices

Addition, multiplication, transpose, inverse $\begin{bmatrix} 4 & -6 \\ -3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{5} & -\frac{3}{5} \\ -\frac{3}{10} & -\frac{2}{5} \end{bmatrix}$

ORDINARY DIFFERENTIAL EQUATIONS

Basic concepts:

Find $y(x)$ where x is the *independent* and y the *dependent variable*, based on an equation involving x, $y(x)$, $y'(x)$, ...e.g.:

$$y''(x) = \frac{1 + y(x) \cdot y'(x)}{1 + x^2}$$

or, equivalently

$$y'' = \frac{1 + yy'}{1 + x^2}$$

The highest derivative (second) is the *order* of the equation.

Solution is normally a *family* of functions, with as many extra *parameters* (C_1, C_2) as the order of the equation.

We will first study *first-order ODE*, then higher-order ODE, but, almost exclusively *linear* (in y and its derivatives) with *constant coefficients*, e.g.: $y''-2y'+3y = e^{2x}$. When the RHS of is zero, the equation is called *homogenous*.

A set (system) of ODE has *several dependent* (unknown) functions y_1, y_2, y_3, ... of a *single independent* variable x. We will study only the *first order, linear* set of ODE with *constant coefficients* (matrix algebra).

Partial differential equations have a *single dependent* variable and *several independent* variables (partial derivatives). We may not have time to discuss these.

First-Order Differential Equations

General form:

$$y' = f(x, y)$$

(visualize graphically). Analytically, this ODE can be solved only in a handful of special cases.

The family of solutions usually (but not always) covers the whole x–y plane by curves which don't intersect (one solution passes through each point). This means: given $y(x_0) = y_0$ (*initial condition*), there is a *unique* solution.

Let's go over the 'special cases' now.

'Trivial' equation:

$$y' = f(x)$$

General solution (if we can integrate f)

$$y(x) = \int f(x)dx + C$$

Example: $y' = \sin(x)$.

Solution: $y(x) = -\cos(x) + C$.

Separable equation

$$y' = h(x) \cdot g(y)$$

Solution**:** Writing y' as $\frac{dy}{dx}$, we can 'separate' x and y:

$$\frac{dy}{g(y)} = h(x)dx$$

and integrate each side *individually* (w/r to y and x, respectively - don't forget to add C). If we can *then* solve for y, we have an *explicit* solution, if not, we leave it in the *implicit* form.

EXAMPLES:

1.

$$
\begin{aligned}
y' &= x \cdot y \\
\frac{dy}{y} &= x\,dx \\
\ln|y| &= \frac{x^2}{2} + \tilde{C} \\
y &= \pm e^{\tilde{C}} \cdot e^{\frac{x^2}{2}} \equiv C e^{\frac{x^2}{2}}
\end{aligned}
$$

2.

$$
\begin{aligned}
9yy' + 4x &= 0 \\
9y\,dy &= -4x\,dx \\
9\frac{y^2}{2} &= -4\frac{x^2}{2} + \tilde{C} \\
y^2 + \frac{4}{9}x^2 &= C
\end{aligned}
$$

family of ellipses centered on the origin, with the vertical versus horizontal diameter in the 2:3 ratio.

3.

$$
\begin{aligned}
y' &= -2xy \\
\frac{dy}{y} &= -2x\,dx \\
\ln|y| &= -x^2 + \tilde{C} \\
y &= Ce^{-x^2}
\end{aligned}
$$

4.

$$(1 + x^2)y' + 1 + y^2 = 0$$

with $y(0) = 1$ (initial value problem).

Solution:

$$\begin{aligned}
\frac{dy}{1+y^2} &= -\frac{dx}{1+x^2} \\
\arctan(y) &= -\arctan(x) + \tilde{C} \\
&\equiv \arctan(C) - \arctan(x) \\
y &= \tan\left(\arctan(C) - \arctan(x)\right) \\
&= \frac{C-x}{1+Cx}
\end{aligned}$$

To find C we solve $1 = \frac{C-0}{1+C \times 0} \Rightarrow C = 1$.

Answer: $y(x) = \frac{1-x}{1+x}$.

Check: $(1+x^2)\frac{d}{dx}\left(\frac{1-x}{1+x}\right) + 1 + \left(\frac{1-x}{1+x}\right)^2 = 0$.

Scale-independent equation

$$y' = g\left(\frac{y}{x}\right)$$

(RHS invariant under $x \to ax$ and $y \to ay$).

Solve by introducing a *new dependent* variable $u(x) = \dfrac{y(x)}{x} \Rightarrow y(x) = x \cdot u(x)$ $\Rightarrow y' = u + xu'$.

Substitute into original equation:

$$xu' = g(u) - u$$

which is *separable* in x and u:

$$\frac{du}{g(u)-u} = \frac{dx}{x}$$

Solve as such, and then go back to $y(x)$.

EXAMPLES:

1.

$$\begin{aligned}
2xyy' - y^2 + x^2 &= 0 \\
y' &= \frac{y}{2x} - \frac{x}{2y} \\
xu' &= -\frac{u^2+1}{2u} \\
\frac{2u\,du}{u^2+1} &= -\frac{dx}{x} \\
\ln(1+u^2) &= -\ln|x| + \tilde{C} \\
u^2+1 &= \frac{2C}{x} \\
y^2 + x^2 - 2Cx &= 0 \\
y^2 + (x-C)^2 &= C^2
\end{aligned}$$

Family of circles having a center at any point of the x-axis, and being tangent to the y-axis

2.

$$\begin{aligned}
x^2y' &= y^2 + xy + x^2 \\
y' &= \left(\frac{y}{x}\right)^2 + \frac{y}{x} + 1 \\
xu' &= u^2 + 1 \\
\frac{du}{1+u^2} &= \frac{dx}{x} \\
\arctan(u) &= \ln|x| + C \\
u &= \tan\left(\ln|x| + C\right) \\
y &= x \cdot \tan(\ln|x| + C)
\end{aligned}$$

Modified Scale-Independent

$$y' = \frac{y}{x} + g\left(\frac{y}{x}\right) \cdot h(x)$$

The same substitution yields

$$xu' = g(u) \cdot h(x)$$

which is also *separable*.

EXAMPLE:

$$
\begin{aligned}
y' &= \frac{y}{x} + \frac{2x^3\cos(x^2)}{y} \\
xu' &= \frac{2x^2\cos(x^2)}{u} \\
u\,du &= 2x\cos(x^2)\,dx \\
\frac{u^2}{2} &= \sin(x^2) + \tilde{C} \\
u &= \pm\sqrt{2\sin(x^2) + C} \\
y &= \pm x\sqrt{2\sin(x^2) + C}
\end{aligned}
$$

Any Other Smart Substitution
(usually suggested), which makes the equation separable.

EXAMPLES:

1.

$$(2x - 4y + 5)y' + x - 2y + 3 = 0$$

Suggestion: introduce: $v = x - 2y$, i.e. $y = \dfrac{x - v}{2}$ and $y' = \dfrac{1 - v'}{2}$

$$
\begin{aligned}
(2v + 5)\frac{1 - v'}{2} + v + 3 &= 0 \\
-(v + \frac{5}{2})v' + 2v + \frac{11}{2} &= 0 \\
\frac{v + \frac{5}{2}}{v + \frac{11}{4}}\,dv &= 2\,dx \\
\left(1 - \frac{\frac{1}{4}}{v + \frac{11}{4}}\right)dv &= 2\,dx \\
v - \frac{1}{4}\ln|v + \frac{11}{4}| &= 2x + C \\
x - 2y - \frac{1}{4}\ln|x - 2y + \frac{11}{4}| &= 2x + C
\end{aligned}
$$

2.

$$y' \cos y + x \sin y = 2x$$

seems to suggest $v = \sin y$, since $v' = y' \cos y$.

The new equation is thus simply

$$v' + xv = 2x$$

which is *separable* and can be solved as such:

$$\begin{aligned} \frac{dv}{v-2} &= -x\,dx \\ \ln|v-2| &= -\frac{x^2}{2} + \ln C \\ v - 2 &= Ce^{-\frac{x^2}{2}} \\ v &= 2 + Ce^{-\frac{x^2}{2}} \end{aligned}$$

Finally, $y = \arcsin v = \arcsin\left(2 + Ce^{-\frac{x^2}{2}}\right)$.

Linear equation

$$y' + g(x) \cdot y = r(x)$$

The solution is constructed in two stages:

1. Solve the *homogeneous* part $y' = -g(x) \cdot y$, which is *separable*, thus:

$$y_h(x) = c \cdot e^{-\int g(x)dx}$$

2. Assume c to be a *function* of x, substitute $c(x) \cdot e^{-\int g(x)dx}$ back into the full equation, and solve the resulting trivial differential equation for $c(x)$.

EXAMPLES:

1.

$$y' + \frac{y}{x} = \frac{\sin x}{x}$$

First solve

$$\begin{aligned}
y' + \frac{y}{x} &= 0 \\
\frac{dy}{y} &= -\frac{dx}{x} \\
\ln|y| &= -\ln|x| + \tilde{c} \\
y &= \frac{c}{x}
\end{aligned}$$

Now substitute this to the original equation:

$$\begin{aligned}
\frac{c'}{x} - \frac{c}{x^2} + \frac{c}{x^2} &= \frac{\sin x}{x} \\
c' &= \sin x \\
c(x) &= -\cos x + C \\
y(x) &= -\frac{\cos x}{x} + \frac{C}{x}
\end{aligned}$$

Note that the solution has always the form of $y_p(x) + Cy_h(x)$, where $y_p(x)$ is a *particular* solution to the full equation, and $y_h(x)$ solves the homogeneous equation only.

Let us verify the former:

$$\frac{d}{dx}\left(-\frac{\cos x}{x}\right) - \frac{\cos x}{x^2} = \frac{\sin x}{x}$$

2.

$$y' - y = e^{2x}$$

First

$$\begin{aligned}
y' - y &= 0 \\
\frac{dy}{y} &= dx \\
y &= ce^x
\end{aligned}$$

Substitute:

$$\begin{aligned}
c'e^x + ce^x - ce^x &= e^{2x} \\
c' &= e^x \\
c(x) &= e^x + C \\
y(x) &= e^{2x} + Ce^x
\end{aligned}$$

3.

$$xy' + y + 4 = 0$$

Homogeneous part:

$$\begin{aligned}
\frac{dy}{y} &= -\frac{dx}{x} \\
\ln|y| &= -\ln|x| + c \\
y &= \frac{c}{x}
\end{aligned}$$

Substitute:

$$\begin{aligned}
c' - \frac{c}{x} + \frac{c}{x} &= -4 \\
c(x) &= -4x + C \\
y(x) &= -4 + \frac{C}{x}
\end{aligned}$$

4.

$$y' + y \cdot \tan(x) = \sin(2x)$$

with $y(0) = 1$.

Homogeneous part:

$$\begin{aligned}
\frac{dy}{y} &= \frac{-\sin x\ dx}{\cos x} \\
\ln|y| &= \ln|\cos x| + \tilde{c} \\
y &= c \cdot \cos x
\end{aligned}$$

Substitute:

$$\begin{aligned}
c' \cos x - c \sin x + c \sin x &= 2 \sin x \cos x \\
c' &= 2 \sin x \\
c(x) &= -2 \cos x + C \\
y(x) &= -2 \cos^2 x + C \cos x
\end{aligned}$$

To find the value of C, solve:

$$1 = -2 + C \Rightarrow C = 3$$

The final answer is thus:

$$y(x) = -2\cos^2 x + 3\cos x$$

To verify:

$$\begin{aligned}&\frac{d}{dx}\left[-2\cos^2 x + 3\cos x\right] + \\ &\left[-2\cos^2 x + 3\cos x\right] \cdot \frac{\sin x}{\cos x} = 2\cos x \sin x\end{aligned}$$

5.

$$x^2 y' + 2xy - x + 1 = 0$$

with $y(1) = 0$.

Homogeneous part:

$$\begin{aligned}\frac{dy}{y} &= -2\frac{dx}{x} \\ \ln|y| &= -2\ln|x| + \tilde{C} \\ y &= \frac{c}{x^2}\end{aligned}$$

Substitute:

$$\begin{aligned}c' - \frac{2c}{x^3} + \frac{2c}{x^3} - x + 1 &= 0 \\ c' &= x - 1 \\ c &= \frac{x^2}{2} - x + C \\ y &= \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}\end{aligned}$$

To meet the initial-value condition:

$$0 = \frac{1}{2} - 1 + C \Rightarrow C = \frac{1}{2}$$

Final answer:

$$y = \frac{(1-x)^2}{2x^2}$$

Verify:

$$x^2 \frac{d}{dx}\left(\frac{(1-x)^2}{2x^2}\right) + 2x\left(\frac{(1-x)^2}{2x^2}\right) - x + 1 \equiv 0.$$

6.

$$y' - \frac{2y}{x} = x^2\cos(3x)$$

First:

$$\begin{aligned} \frac{dy}{y} &= 2\frac{dx}{x} \\ \ln|y| &= 2\ln|x| + \tilde{c} \\ y &= cx^2 \end{aligned}$$

Substitute:

$$\begin{aligned} c'x^2 + 2cx - 2cx &= x^2\cos(3x) \\ c' &= \cos(3x) \\ c &= \frac{\sin(3x)}{3} + C \\ y &= \frac{x^2}{3}\sin(3x) + Cx^2 \end{aligned}$$

To verify the *particular* solution:

$$\frac{d}{dx}\left(\frac{x^2}{3}\sin(3x)\right) - \frac{2x}{3}\sin(3x) = x^2\cos(3x)$$

Bernoulli equation

$$y' + f(x)\cdot y = r(x)\cdot y^a$$

where a is a specific, constant exponent.

Introducing a *new dependent* variable $u = y^{1-a}$, i.e. $y = u^{\frac{1}{1-a}}$, one gets:

$$\frac{1}{1-a}u^{\frac{1}{1-a}-1}u' + f(x)\cdot u^{\frac{1}{1-a}} = r(x)\cdot u^{\frac{a}{1-a}}$$

Multiplying by $(1-a)u^{-\frac{a}{1-a}}$ results in:

$$u' + (1-a)f(x) \cdot u = (1-a)r(x)$$

which is *linear* in u' and u.

The answer is then easily converted back to $y = u^{\frac{1}{1-a}}$.

EXAMPLES:

1.
$$y' + xy = \frac{x}{y}$$

Bernoulli, $a = -1,\ f(x) \equiv x,\ g(x) \equiv x$, implying

$$u' + 2xu = 2x$$

where $y = u^{\frac{1}{2}}$.

Solving as linear:

$$\begin{aligned} \frac{du}{u} &= -2x\,dx \\ \ln|u| &= -x^2 + \tilde{c} \\ u &= c \cdot e^{-x^2} \end{aligned}$$

Substitute:

$$\begin{aligned} c'e^{-x^2} - 2xce^{-x^2} + 2xce^{-x^2} &= 2x \\ c' &= 2xe^{x^2} \\ c(x) &= e^{x^2} + C \\ u(x) &= 1 + Ce^{-x^2} \\ y(x) &= \pm\sqrt{1 + Ce^{-x^2}} \end{aligned}$$

(one can easily check that this is a solution with either the $+$ or the $-$ sign).

2.
$$2xy' = 10x^3y^5 + y$$

(terms reshuffled a bit).

Bernoulli with $a = 5$, $f(x) = -\frac{1}{2x}$, and $g(x) = 5x^2$

This implies

$$u' + \frac{2}{x}u = -20x^2$$

with $y = u^{-\frac{1}{4}}$.

Solving as linear:

$$\begin{aligned} \frac{du}{u} &= -2\frac{dx}{x} \\ \ln|u| &= -2\ln|x| + \tilde{c} \\ u &= \frac{c}{x^2} \end{aligned}$$

Substituted back into the full equation:

$$\begin{aligned} \frac{c'}{x^2} - 2\frac{c}{x^3} + 2\frac{c}{x^3} &= -20x^2 \\ c' &= -20x^4 \\ c(x) &= -4x^5 + C \\ u(x) &= -4x^3 + \frac{C}{x^2} \\ y(x) &= \pm\left(-4x^3 + \frac{C}{x^2}\right)^{-\frac{1}{4}}. \end{aligned}$$

3.

$$2xyy' + (x-1)y^2 = x^2e^x$$

Bernoulli with $a = -1$, $f(x) = \frac{x-1}{2x}$, and $g(x) = \frac{x}{2}e^x$

This translates to:

$$u' + \frac{x-1}{x}u = xe^x$$

with $y = u^{\frac{1}{2}}$.

Solving homogeneous part:

$$\begin{aligned} \frac{du}{u} &= (\frac{1}{x} - 1)\,dx \\ \ln|u| &= \ln|x| - x + \tilde{c} \\ u &= cxe^{-x} \end{aligned}$$

Substituted:

$$\begin{aligned} & c'xe^{-x} + ce^{-x} - cxe^{-x} + (x-1)ce^{-x} \\ = \; & xe^{x} \\ c' = \; & e^{2x} \\ c(x) = \; & \frac{1}{2}e^{2x} + C \\ u(x) = \; & \frac{x}{2}e^{x} + Cxe^{-x} \\ y(x) = \; & \pm\sqrt{\frac{x}{2}e^{x} + Cxe^{-x}} \end{aligned}$$

Exact equation

General idea:
Suppose we have a function of x and y, $f(x,y)$ say.
Then

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

is its *total differential*, representing the function's increase from (x,y) to $(x+dx, y+dy)$.

By making this equal to zero (a differential equation, called *exact*), we are effectively saying that $f(x,y) = C$ (and this is its implicit solution).

EXAMPLE: Suppose

$$f(x,y) = x^2y - 2x$$

This means that

$$(2xy-2)\,dx + x^2\,dy = 0$$

has a simple solution

$$\begin{aligned} x^2y - 2x &= C \\ y &= \frac{2}{x} + \frac{C}{x^2} \end{aligned}$$

Note that the differential equation can be also written as:

$$y' = 2\frac{1-xy}{x^2}$$

(linear).

We must now try to *reverse* the process, i.e. given a *differential equation*, *find* $f(x,y)$.

There are then *two issues* to be settled:

1. How do we know that an equation *is* exact?
2. Knowing it is, how do we solve it?

To answer the first question, we recall that

$$\frac{\partial^2 f}{\partial x\,\partial y} \equiv \frac{\partial^2 f}{\partial y\,\partial x}$$

Thus, $g(x,y)\,dx + h(x,y)\,dy = 0$ is exact *if and only if*

$$\frac{\partial g}{\partial y} \equiv \frac{\partial h}{\partial x}$$

As to *solving* the equation, we proceed in three stages:

1. Find
$$G(x,y) = \int g(x,y)\,dx$$
(considering y a constant).
2. Construct
$$H(y) = h(x,y) - \frac{\partial G}{\partial y}$$
[must be a function of y only, as $\frac{\partial H}{\partial x} = \frac{\partial h}{\partial x} - \frac{\partial^2 G}{\partial x\,\partial y} = \frac{\partial g}{\partial y} - \frac{\partial g}{\partial y} \equiv 0$].
3. $$f(x,y) = G(x,y) + \int H(y)\,dy$$
Proof: $\frac{\partial f}{\partial x} = \frac{\partial G}{\partial x} = g$ and $\frac{\partial f}{\partial y} = \frac{\partial G}{\partial y} + H = h$.

EXAMPLE:

$$2x\sin(3y)\,dx + \big(3x^2\cos(3y) + 2y\big)\,dy = 0$$

Let us first verify that the equation *is* exact:

$$\frac{\partial}{\partial y} 2x \sin(3y) = 6x \cos 3y$$

$$\frac{\partial}{\partial x} \left(3x^2 \cos(3y) + 2y\right) = 6x \cos 3y$$

Solving it:

$$\begin{aligned} G &= x^2 \sin(3y) \\ H &= 3x^2 \cos(3y) + 2y - 3x^2 \cos(3y) = 2y \\ f(x, y) &= x^2 \sin(3y) + y^2 \end{aligned}$$

Answer: $y^2 + x^2 \sin(3y) = C$ (implicit form).

Integrating Factors

Any first-order ODE (e.g. $y' = \frac{y}{x}$) can be expanded to make it *look like* an exact equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x} \\ y\,dx - x\,dy &= 0 \end{aligned}$$

But since $\frac{\partial(y)}{\partial y} \neq -\frac{\partial(x)}{\partial x}$, this equation is *not* exact.

The good news is that, theoretically, there is always a function of x and y, say $F(x, y)$, which can multiply the equation to make it exact. This function is called an *integrating factor*.

The bad news is that there is no general procedure for finding $F(x, y)$.

Yet, there are *two special cases* when it *is* possible:
Let us write the differential equation in its 'look-like-exact' form of

$$P(x, y)dx + Q(x, y)dy = 0$$

where $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$. One can find an integrating factor from

1. $$\frac{d \ln F}{dx} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}$$

iff the right hand side of this equation is a function of x *only*

Proof: $FPdx + FQdy = 0$ is exact when

$$\frac{\partial(FP)}{\partial y} = \frac{\partial(FQ)}{\partial x}$$

which is the same as

$$F\frac{\partial P}{\partial y} = \frac{dF}{dx}Q + F\frac{\partial Q}{\partial x}$$

assuming that F is a function of x *only*. Solving for $\frac{\frac{dF}{dx}}{F}$ results in $\frac{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}{Q}$. When the last expression contains no y, we simply integrate it (with respect to x) to find $\ln F$.

2. or from

$$\frac{d\ln F}{dy} = \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P}$$

iff the right hand side is a function of y *only*.

EXAMPLES:

1. Let us try solving our $y\,dx - x\,dy = 0$. Since

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = -\frac{2}{x}$$

we have

$$\ln F = -2\int \frac{dx}{x} = -2\ln x$$

(no need to bother with a constant) $\Rightarrow F = \frac{1}{x^2}$. Thus $\frac{y}{x^2}\,dx - \frac{1}{x}\,dy = 0$ must be exact (check it). Solving it gives $-\frac{y}{x} = \tilde{C}$, or $y = Cx$.

Note that there is infinitely many integrating factors, if $F(x,y)$ is one, so is $F(x,y)\cdot R(f(x,y))$, where R is an *arbitrary* function.

2.

$$\left(2\cos y + 4x^2\right)\,dx = x\sin y\,dy$$

Since

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{-2\sin y + \sin y}{-x\sin y} = \frac{1}{x}$$

we get

$$\begin{aligned}\ln F &= \int \frac{1}{x}\, dx = \ln x \\ F &= x.\end{aligned}$$

$$\left(2x\cos y + 4x^3\right)\, dx - x^2 \sin y\, dy = 0$$

is therefore exact, and can be solved as such:

$$\begin{aligned} & x^2\cos y + x^4 = C \\ y &= \arccos\left(\frac{C}{x^2} - x^2\right)\end{aligned}$$

3.

$$(3xe^y + 2y)\, dx + (x^2e^y + x)\, dy = 0$$

Trying again

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{xe^y + 1}{x^2e^y + x} = \frac{1}{x}$$

which means that

$$\begin{aligned}\ln F &= \int \frac{dx}{x} = \ln x \\ F &= x\end{aligned}$$

.

$$(3x^2e^y + 2xy)\, dx + (x^3e^y + x^2)\, dy = 0$$

is exact. Solving it yields:

$$x^3e^y + x^2y = C$$

Exact equations - more examples:

1)

$$(1 - e^{x-y})dx + (x + 2ye^{-y})dy = 0$$

$$\begin{aligned}
\frac{\partial P}{\partial y} &= e^{x-y} \\
\frac{\partial Q}{\partial x} &= 1 \\
\frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} &= 1 \\
\ln F &= y \\
F &= e^y
\end{aligned}$$

$$(e^y - e^x)dx + (xe^y + 2y)dy = 0$$

$$\begin{aligned}
G &= xe^y - x \\
H &= xe^y + 2y - xe^y = 2y \\
f &= xe^y - x + y^2 = C
\end{aligned}$$

2)

$$(2x + \tan y)dx + [x - (1 + x^2)\tan y]dy = 0$$

$$\begin{aligned}
\frac{\partial P}{\partial y} &= \frac{\sin^2 y + \cos^2 y}{\cos^2 y} = \tan^2 y + 1 \\
\frac{\partial Q}{\partial x} &= 1 - 2x\tan y \\
\frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} &= -\frac{2x\tan y + \tan^2 y}{2x + \tan y} = -\tan y \\
\ln F &= \ln(\cos y)
\end{aligned}$$

$$(2x\cos y + \sin y)dx + [x\cos y - (1 + x^2)\sin y]dy = 0$$

$$\begin{aligned}
G &= x^2\cos y + x\sin y \\
H &= x\cos y - (1 + x^2)\sin y + x^2\sin y - x\cos y = -\sin y \\
f &= x^2\cos y + x\sin y + \cos y = C
\end{aligned}$$

There are many other special types of equations (usually, of a more 'exotic' type), we will look at one example only:

Clairaut equation:

$$y = xy' + g(y')$$

where g is an arbitrary function. The idea is to introduce $p \equiv y'$, differentiate the original equation with respect to x, obtaining

$$\begin{aligned} p &= p + xp' + p'g'(p) \\ & p' \cdot (x + g'(p)) = 0 \end{aligned}$$

This implies that either $p \equiv y' = C \Rightarrow$

$$y = xC + g(C)$$

which represents a family of *regular* solutions (all straight lines), or

$$x = -g'(p)$$

which, when solved for p and substituted back into $y = xp + g(p)$ provides the so called *singular* solution (an *envelope* of the regular family).

EXAMPLE:

$$(y')^2 - xy' + y = 0$$

(terms reshuffled a bit) is solved by either

$$y = Cx - C^2$$

or

$$\begin{aligned} x &= 2p \\ p &= \frac{x}{2} \\ y &= xp - p^2 = \frac{x^2}{4} \end{aligned}$$

(singular solution).

Note that for an initial condition below or at the parabola two possible solutions exist, above the parabola there is none.

A final stratagem:

When an equation appears more complicated in terms of y rather than x, e.g.

$$(2x + y^4)y' = y$$

one can try reversing the role of x and y. All it takes is to replace $y' \equiv \frac{dy}{dx}$ by $\dfrac{1}{\frac{dx}{dy}}$, for example (using the previous equation):

$$\frac{dx}{dy} = 2\frac{x}{y} + y^3$$

It's a pain to remember that x is now the dependent and y is the independent variable, so what I like to do is $x \leftrightarrow y$:

$$y' = 2\frac{y}{x} + x^3$$

The last equation is linear and can be solved as such:

$$\begin{aligned} \frac{dy}{y} &= 2\frac{dx}{x} \\ \ln|y| &= 2\ln|x| + \ln c \\ y &= c(x) \cdot x^2 \end{aligned}$$

Substituted into the full equation:

$$\begin{aligned} 2xc + x^2c' &= 2\frac{x^2c}{x} + x^3 \\ c' &= x \\ c &= \frac{x^2}{2} - C \\ y &= \frac{x^4}{2} - Cx^2 \end{aligned}$$

And only now I would go back to:

$$x = \frac{y^4}{2} - Cy^2$$

which can be solved explicitly for $y = \pm\sqrt{C \pm \sqrt{C^2 + 2x}}$.

Applications

Of Geometric Kind:

1. Find a curve such that, from each of its points, the distance to the origin is the same as the distance to the intersection of its *normal* (i.e. perpendicular straight line) with the x-axis.

   Solution: Suppose $y(x)$ is the equation of the curve (yet unknown). The equation of the normal is

   $$Y - y = -\frac{1}{y'} \cdot (X - x)$$

   where (x, y) [fixed] are the points of the curve, and (X, Y) [variable] are the points of the normal. This normal intersects the x-axis at $Y = 0$ and $X = yy' + x$. The distance between this and the original (x, y) is

   $$\sqrt{(yy')^2 + y^2}$$

   the distance from (x, y) to $(0, 0)$ is

   $$\sqrt{x^2 + y^2}$$

   These two distances are equal when

   $$y^2 (y')^2 = x^2$$

   or

   $$y' = \pm \frac{x}{y}$$

   This is a separable differential equation easy to solve:

   $$y^2 \pm x^2 = C$$

   The curves are either circles centered on $(0, 0)$, or hyperbolas [with $y = \pm x$ as special cases].

2. Find a curve whose normals (all) pass through the origin.

Solution (we can guess the answer, but let us do it properly): Into the same equation of the curve's normal (see above), we substitute 0 for both X and Y, since the straight line must pass through $(0,0)$. This gives:

$$-y = \frac{x}{y'}$$

which is simple to solve:

$$\begin{aligned} -y\,dy &= x\,dx \\ x^2 + y^2 &= C \end{aligned}$$

(circles centered on the origin – we knew that!).

3. A family of curves covering the whole x–y plane enables one to draw lines perpendicular to these curves. The collection of all such lines is yet another family of curves *orthogonal* (i.e. perpendicular) to the original family. If we can find the differential equation $y' = f(x,y)$ having the original family of curves as its solution, we can find the corresponding orthogonal family by solving $y' = -\frac{1}{f(x,y)}$. The next set of examples relates to this.

   1. The original family is described by

      $$x^2 + (y - C)^2 = C^2$$

      with C arbitrary (i.e. collection of circles tangent to the x-axis at the origin). To find the corresponding differential equation, we differentiate the original equation with respect to x:

      $$2x + 2(y - C)y' = 0$$

      solve for $y' = \frac{x}{C-y}$, and then eliminate C by solving the original equation for C, thus:

      $$\begin{aligned} & x^2 + y^2 - 2Cy = 0 \\ C &= \frac{x^2 + y^2}{2y} \end{aligned}$$

      further implying

      $$y' = \frac{x}{\frac{x^2+y^2}{2y} - y} = \frac{2xy}{x^2 - y^2}$$

To find the orthogonal family, we solve

$$y' = \frac{y^2 - x^2}{2xy}$$

[scale-independent equation solved earlier]. The answer was:

$$(x - C)^2 + y^2 = C^2$$

i.e. collection of circles tangent to the y-axis at the origin.

2. Let the original family be circles centered on the origin (it should be clear what the orthogonal family is, but again, let's solve it anyhow):

$$x^2 + y^2 = C^2$$

describes the original family,

$$2x + 2yy' = 0$$

is the corresponding differential equation, equivalent to

$$y' = -\frac{x}{y}$$

The orthogonal family is the solution to

$$\begin{aligned} y' &= \frac{y}{x} \\ \frac{dy}{y} &= \frac{dx}{x} \\ y &= Cx \end{aligned}$$

(straight lines passing through the origin).

3. Let the original family be described by

$$y^2 = x + C$$

(the $y^2 = x$ parabola slid horizontally). The corresponding differential equation is

$$2yy' = 1$$

the 'orthogonal' equation:

$$y' = -2y.$$

Answer:

$$\begin{aligned} \ln|y| &= -2x + \tilde{C} \\ y &= Ce^{-2x} \end{aligned}$$

(try to visualize the curves).

4. Finally, let us start with

$$y = Cx^2$$

(all parabolas tangent to the x-axis at the origin). Differentiating:

$$y' = 2Cx$$

implying (since $C = \frac{y}{x^2}$)

$$y' = 2\frac{y}{x}$$

The 'orthogonal' equation is

$$\begin{aligned} y' &= -\frac{x}{2y} \\ y^2 + \frac{x^2}{2} &= C \end{aligned}$$

(collection of ellipses centered on the origin, with the x-diameter being $\sqrt{2}$ times bigger than the y-diameter).

4. The position of four ships on the ocean is such that the ships form vertices of a square of length L. At the same instant each ship fires a missile that directs its motion towards the missile on its right. Assuming that the four missiles fly horizontally and with the same constant speed, find the path of each.

In Physics:

If a hole is made at a bottom of a container, water will flow out at the rate of $a\sqrt{h}$, where a depends on the size of the opening (we will keep that constant) and h is the height of the (remaining) water, which varies in time. Time t is the independent variable. Find $h(t)$ as a function of t for:

1. A *cylindrical* container of radius r and height h_0.

   Solution: First we have to establish the *volume* V of the remaining water as a function of height. In this case we get simply

$$V(h) = \pi r^2 h$$

   Differentiating with respect to t we get:

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

   This in turn must be equal to $-a\sqrt{h}$, since the rate at which the water is flowing out must be equal to the rate at which its volume is *decreasing*. Thus

$$\pi r^2 \dot{h} = -a\sqrt{h}$$

   where $\dot{h} \equiv \frac{dh}{dt}$. This is a simple (separable) differential equation, which we solve by

$$\begin{aligned} \frac{dh}{\sqrt{h}} &= -\frac{a}{\pi r^2} dt \\ \frac{h^{1/2}}{\frac{1}{2}} &= -\frac{at}{\pi r^2} + C \\ \sqrt{h} &= -\frac{at}{2\pi r^2} + \sqrt{h_0} \end{aligned}$$

   or equivalently

$$t = \frac{2\pi r^2}{a}\left(\sqrt{h_0} - \sqrt{h}\right)$$

   Subsidiary: What percentage of time is spent emptying the last 20% of the container?

   Solution:

$$t_{100} = \frac{2\pi r^2}{a}\sqrt{h_0}$$

   is the time to fully empty the container.

$$t_{80} = \frac{2\pi r^2}{a}\left(\sqrt{h_0} - \sqrt{\frac{h_0}{5}}\right)$$

is the time it takes to empty the first 80% of the container. The answer:

$$\frac{t_{100} - t_{80}}{t_{100}} = \sqrt{\frac{1}{5}} = 44.72\%$$

2. A *conical* container with the top radius (at h_0) equal to r.

   Solution:

$$V(h) = \frac{1}{3}\pi h \left(r \frac{h}{h_0} \right)^2$$

(Note that one fifth of the full *volume* corresponds to $\left(\frac{1}{5}\right)^{1/3} h_0$, i.e. 58.48% of the full *height*!). Thus

$$\frac{2\pi r^2}{3h_0^2} h^2 \dot{h} = -a\sqrt{h}$$

is again a separable equation:

$$\begin{aligned}
h^{3/2}\, dh &= -\frac{3ah_0^2}{2\pi r^2}\, dt \\
h^{5/2} &= -\frac{15ah_0^2}{4\pi r^2}\, t + h_0^{5/2} \\
t &= \frac{4\pi r^2}{15ah_0^2}\left(h_0^{5/2} - h^{5/2}\right)
\end{aligned}$$

This implies

$$t_{100} = \frac{4\pi r^2 \sqrt{h_0}}{15a}$$

and

$$t_{80} = \frac{4\pi r^2 \sqrt{h_0}}{15a} \cdot \left[1 - \left(\frac{1}{5}\right)^{5/6} \right]$$

implying

$$\frac{t_{100} - t_{88}}{t_{100}} = \left(\frac{1}{5}\right)^{5/6} = 26.15\%$$

3. A *hemisphere* of radius R.

   Solution:

$$V(h) = \frac{1}{3}\pi h^2 (3R - h)$$

Making the right hand side equal to $\frac{2}{3}\pi R^3/5$ and solving for h gives the height of the 20% (remaining) volume to equal $0.391600R$. Now

$$\begin{aligned}\left[\frac{2\pi}{3}h(3R-h)-\frac{1}{3}\pi h^2\right]\overset{\bullet}{h} &= \\ \left(2\pi hR-\pi h^2\right)\overset{\bullet}{h} &= -a\sqrt{h}\end{aligned}$$

which can be easily solved, thus:

$$\left(\frac{4\pi}{3}Rh^{3/2}-\frac{2\pi}{5}h^{5/2}\right)-\frac{14\pi}{15}R^{5/2}=-a\cdot t$$

Thus,

$$t_{100}=\frac{14\pi}{15a}R^{5/2}$$

and

$$t_{80}=\frac{14\pi}{15a}R^{5/2}-\left(\frac{4\pi}{3a}R^{5/2}\times 0.3916^{3/2}-\frac{2\pi}{5a}R^{5/2}\times 0.3916^{5/2}\right)$$

which implies

$$\frac{t_{100}-t_{88}}{t_{100}}=\frac{10}{7}\times 0.3916^{3/2}-\frac{3}{7}\times 0.3916^{5/2}=30.9\%$$

Second Order Differential Equations

Much more difficult to solve than first-order ODEs - we will concentrate mainly on the simplest case of *linear* equations with *constant coefficients*.

The general solution always has two arbitrary constants, say C_1 and C_2, which means that we need **two conditions** to pull out a unique solution.

These are usually of two distinct types:

1. initial conditions:

$$\begin{aligned}y(x_0) &= a \\ y'(x_0) &= b\end{aligned}$$

[x_0 is often 0], specifying the *value* and *slope* of the function at a point,

2. boundary conditions:

$$\begin{aligned} y(x_1) &= a \\ y(x_2) &= b \end{aligned}$$

specifying a value each at two distinct points.

In this introductory section we look at *two* special **non-linear** cases, both **reducible to first order.**

i) missing y

(i.e. does not appear explicitly - only x, y' and y'' do), then $y' \equiv z$ can be considered the unknown function of the equation. In terms of $z(x)$, the equation is of the *first* order.

Once we solve for $z(x)$, just integrate the result to get $y(x)$.

1.

$$\begin{aligned} y'' &= y' \\ z' &= z \\ \frac{dz}{z} &= dx \\ \ln|z| &= x + \ln C_1 \\ z &= C_1 e^x \\ y &= C_1 e^x + C_2 \end{aligned}$$

Let's impose the following initial conditions: $y(0) = 0$ and $y'(0) = 1$.

By substituting into the general solution we get:

$$\begin{aligned} C_1 + C_2 &= 0 \\ C_1 &= 1 \end{aligned}$$

implying $C_2 = -1$ and

$$y = e^x - 1$$

2.

$$
\begin{aligned}
xy'' + y' &= 0 \\
xz' + z &= 0 \\
\frac{dz}{z} &= -\frac{dx}{x} \\
\ln|z| &= \ln|x| + \ln C_1 \\
z &= \frac{C_1}{x} \\
y &= C_1 \ln|x| + C_2
\end{aligned}
$$

Let us make this into a boundary-value problem: $y(1) = 1$ and $y(3) = 0$ implying

$$
\begin{aligned}
C_2 &= 1 \\
C_1 \ln 3 + C_2 &= 0
\end{aligned}
$$

i.e. $C_1 = -\frac{1}{\ln 3}$ and

$$y = 1 - \frac{\ln|x|}{\ln 3}$$

3.

$$
\begin{aligned}
xy'' + 2y' &= 0 \\
xz' + 2z &= 0 \\
\frac{dz}{z} &= -2\frac{dx}{x} \\
z &= \frac{\tilde{C}_1}{x^2} \\
y &= \frac{C_1}{x} + C_2
\end{aligned}
$$

Sometimes the two extra conditions can be of a more bizarre type: $y(2) = \frac{1}{2}$, implying

$$\frac{1}{2} = \frac{C_1}{2} + C_2$$

and requiring that the solution intersects the $y = x$ straight line at the right angle.

Translated into our notation: $y'(x_0) = -1$ where x_0 is a solution to $y(x) = x$, i.e.

$$-\frac{C_1}{x_0^2} = -1$$

where

$$\frac{C_1}{x_0} + C_2 = x_0$$

Solution: $C_2 = 0$, $C_1 = 1$ and $x_0 = 1$. The final answer:

$$y(x) = \frac{1}{x}$$

The second type (of a second-order equation reducible to first order) has ii) missing x (not appearing explicitly) such as

$$y \cdot y'' + (y')^2 = 0$$

We again introduce $z \equiv y'$ as a new dependent variable, but this time we see it as a function of y, which becomes the *independent* variable of the new equation!

Since $y'' = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$ [chain rule], we replace y'' in the original equation by $\frac{dz}{dy} \cdot z$.

We solve the resulting first-order equation for z (as a function of y), then replace z by y' and solve again.

1.

$$\begin{aligned}
y \cdot y'' + (y')^2 &= 0 \\
y\frac{dz}{dy}z + z^2 &= 0 \\
\frac{dz}{z} &= -\frac{dy}{y} \\
\ln|z| &= -\ln|y| + \ln C_1 \\
z &= \frac{\tilde{C}_1}{y} \\
y' &= \frac{\tilde{C}_1}{y} \\
y\,dy &= \tilde{C}_1 dx \\
y^2 &= C_1 x + C_2 \\
y &= \pm\sqrt{C_1 x + C_2}
\end{aligned}$$

(note that $z = 0$ solution is included).

2.

$$\begin{aligned}
y'' + e^{2y}(y')^3 &= 0 \\
\frac{dz}{dy}z + e^{2y}z^3 &= 0 \\
\frac{dz}{z^2} &= -e^{2y}dy \\
-\frac{1}{z} &= -\frac{1}{2}e^{2y} - C_1 \\
z &= \frac{1}{C_1 + \frac{1}{2}e^{2y}} \\
(C_1 + \frac{1}{2}e^{2y})dy &= dx \\
C_1 y + \frac{1}{4}e^{2y} &= x + C_2
\end{aligned}$$

This time, we need to add: $y = C$.

3.

$$\begin{aligned}
y'' + (1 + \frac{1}{y})(y')^2 &= 0 \\
\frac{dz}{dy}z + (1 + \frac{1}{y})z^2 &= 0 \\
\frac{dz}{z} &= -(1 + \frac{1}{y})dy \\
\ln|z| &= -\ln|y| - y + \ln C_1 \\
z &= \frac{C_1}{y}e^{-y} \\
ye^y dy &= C_1 dx \\
(y-1)e^y &= C_1 x + C_2
\end{aligned}$$

(covers the $y = C$ case).

LINEAR EQUATION

The most general form is

$$y'' + f(x)y' + g(x)y = r(x)$$

where f, g and r are specific functions of x. When $r \equiv 0$ the equation is called *homogeneous*.

There is no general technique for solving this equation, but some **results** relating to it are worth quoting:

1. The general solution must look like this: $y = C_1 y_1 + C_2 y_2 + y_p$, where y_1 and y_2 are linearly independent 'basic' solutions (if we only knew how to find them!) of the corresponding *homogeneous* equation, and y_p is a *particular* solution to the *full* equation. None of these are unique (e.g. $y_1 + y_2$ and $y_1 - y_2$ is yet another basic set, etc.).

2. When one basic solution (say y_1) of the *homogeneous* version of the equation is known, the other can be found by variation of parameters (VP): Assume that its solution has the form of $c(x)y_1(x)$, substitute this *trial solution* into the equation and get a *first-order* differential equation for $c' \equiv z$.

EXAMPLES:

1. $$y'' - 4xy' + (4x^2 - 2)y = 0$$

   given that

   $$y_1 = \exp(x^2)$$

   is a solution. To verify that, $y_1' = 2x\exp(x^2)$ and $y_1'' = 2\exp(x^2) + 4x^2\exp(x^2)$. Substituting

   $$y = c \cdot \exp(x^2)$$

   back into the equation - remember: $y' = c'y_1 + cy_1'$ and $y'' = c''y_1 + 2c'y_1' + cy_1''$; also remember that the c-proportional terms must *cancel* out - yields

   $$c''\exp(x^2) + 4xc'\exp(x^2) - 4xc'\exp(x^2) = 0$$
   $$c'' = 0$$

This we thus always be the case of 'missing y, now called c, equation':

$$\begin{aligned} z' &= 0 \\ z &= C_1 \\ c &= C_1 x + C_2 \end{aligned}$$

Substituting back to y results in

$$y = C_1 x \exp(x^2) + C_2 \exp(x^2)$$

We can thus identify $x \exp(x^2)$ as our y_2.

2.

$$y'' + \frac{2}{x} y' + y = 0$$

given that

$$y_1 = \frac{\sin x}{x}$$

is a solution. To verify: $y_1' = \frac{\cos x}{x} - \frac{\sin x}{x^2}$ and $y_1'' = -\frac{\sin x}{x} - 2\frac{\cos x}{x^2} + 2\frac{\sin x}{x^3}$.

$$y = c \cdot \frac{\sin x}{x}$$

substituted, yields:

$$\begin{aligned} c''\frac{\sin x}{x} + 2c'(\frac{\cos x}{x} - \frac{\sin x}{x^2}) + \frac{2}{x} c' \frac{\sin x}{x} &= 0 \\ c'' \sin x + 2c' \cos x &= 0 \\ z' \sin x + 2z \cos x &= 0 \\ \frac{dz}{z} &= -2\frac{\cos x\, dx}{\sin x} \\ \ln|z| &= -2 \ln|\sin x| + \ln \tilde{C}_1 \\ z &= \frac{-C_1}{\sin^2 x} \\ c &= C_1 \frac{\cos x}{\sin x} + C_2 \\ y &= C_1 \frac{\cos x}{x} + C_2 \frac{\sin x}{x} \end{aligned}$$

When both basic solutions of the homogeneous version are known, a *particular* solution to the full **non-homogeneous** equations can be also found by an extended VP idea.

This time we have two unknown 'parameters', denoted u and v, instead of the old single c.

We now derive **general formulas** for u and v:

We need to find y_p only, which we take to have the *trial* form of

$$u(x) \cdot y_1 + v(x) \cdot y_2$$

with $u(x)$ and $v(x)$ yet to be found (the variable 'parameters'). Substituting this into the full equation [note that terms proportional to $u(x)$, and those proportional to $v(x)$, cancel out], we get:

$$u''y_1 + 2u'y_1' + v''y_2 + 2v'y_2' + f(x)\,(u'y_1 + v'y_2) = r(x)$$

This is a single differential equation for *two* unknown functions (u and v), which means we are free to impose yet another *arbitrary* constraint on u and v. This is chosen to simplify the previous equation, thus:

$$u'y_1 + v'y_2 = 0$$

which further implies (after one differentiation) that

$$u''y_1 + u'y_1' + v''y_2 + v'y_2' = 0$$

The original (V of P) equation therefore simplifies to

$$u'y_1' + v'y_2' = r(x)$$

The two equations can be solved (*algebraically*, using Cramer's rule) for

$$\begin{aligned} u' &= \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \\ &= \frac{y_2 r}{y_2 y_1' - y_1 y_2'} \end{aligned}$$

and

$$
\begin{aligned}
v' &= \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \\
&= \frac{y_1 r}{y_1 y_2' - y_2 y_1'}
\end{aligned}
$$

where the denominator it called the Wronskian of the two basic solutions (these are linearly independent iff their Wronskian is nonzero; one can use this as a check – useful later when dealing with more than two basic solutions). From the last two expressions, one can easily find u and v by an extra integration (the right hand sides are known functions of x).

EXAMPLE:

$$y'' - 4xy' + (4x^2 - 2)y = 4x^4 - 3$$

We have already solved the homogeneous version getting

$$y_1 = \exp(x^2)$$

and

$$y_2 = x\exp(x^2)$$

Using the previous two formulas we get

$$
\begin{aligned}
u' &= (3 - 4x^4)x\exp(-x^2) \\
u(x) &= (\frac{5}{2} + 4x^2 + 2x^4)\exp(-x^2) + C_1
\end{aligned}
$$

and

$$
\begin{aligned}
v' &= (4x^4 - 3)\exp(-x^2) \\
v(x) &= -(3 + 2x^2)x\exp(-x^2) + C_2
\end{aligned}
$$

Simplifying $uy_1 + vy_2$ yields

$$(\frac{5}{2} + x^2) + C_1\exp(x^2) + C_2 x\exp(x^2)$$

which identifies $\frac{5}{2} + x^2$ as a particular solution of the full equation (this can be verified easily).

Given three specific functions y_1, y_2 and y_p, it is possible to **construct** the corresponding differential **equation** which has $C_1 y_1 + C_2 y_2 + y_p$ as its general solution (that's how I set up exam questions).

EXAMPLE:

Knowing that

$$y = C_1 x^2 + C_2 \ln x + \frac{1}{x}$$

we first substitute x^2 and $\ln x$ for y in

$$y'' + f(x) y' + g(x) y = 0$$

[the homogeneous version] to get:

$$\begin{aligned} 2 + 2x \cdot f + x^2 \cdot g &= 0 \\ -\frac{1}{x^2} + \frac{1}{x} \cdot f + \ln x \cdot g &= 0 \end{aligned}$$

and solve, *algebraically*, for

$$\begin{aligned} \begin{bmatrix} f \\ g \end{bmatrix} &= \begin{bmatrix} 2x & x^2 \\ \frac{1}{x} & \ln x \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ \frac{1}{x^2} \end{bmatrix} \\ f &= \frac{-2\ln x - 1}{x(2\ln x - 1)} \\ g &= \frac{4}{x^2(2\ln x - 1)} \end{aligned}$$

The left hand side of the equation is therefore

$$y'' + \frac{-2\ln x - 1}{x\,(2\ln x - 1)}\, y' + \frac{4}{x^2\,(2\ln x - 1)}\, y$$

[one could multiply the whole equation by $x^2(2\ln x - 1)$ to simplify the answer].

To ensure that $\frac{1}{x}$ is a particular solution, we substitute it into the left hand side of the last equation(for y), yielding $r(x)$ [$= \frac{3\,(2\ln x+1)}{x^3\,(2\ln x-1)}$ in our case]. The final answer is thus:

$$x^2(2\ln x - 1)y'' - x(2\ln x + 1)y' + 4y = \frac{3}{x}(2\ln x + 1)$$

With constant coefficients

From now on we will assume that the two 'coefficients' $f(x)$ and $g(x)$ are x-independent constants, and call them a and b. The equation we want to solve is then

$$y'' + ay' + by = r(x)$$

with a and b being two specific *numbers*. We will start with the

Homogeneous Case $[r(x) \equiv 0]$

All we have to do is to find two linearly *independent* basic solutions y_1 and y_2, and then combine them in the $c_1 y_1 + c_2 y_2$ manner (as we already know from the general case).

To achieve this, we *try* a solution of the following form:

$$y_{\text{trial}} = e^{\lambda x}$$

where λ is a number whose value is yet to be determined. Substituting this into $y'' + ay' + by = 0$ and dividing by $e^{\lambda x}$ results in

$$\lambda^2 + a\lambda + b = 0$$

which is the so called **characteristic polynomial** for λ.

When this (quadratic) equation has two real roots the problem is solved (we have gotten our two basic solutions). What do we do when the two roots are complex, or when only a single root exists? Let us look at these possibilities, one by one.

1. **Two (distinct) real roots**.

   EXAMPLE:

$$\begin{aligned} y'' + y' - 2y &= 0 \\ \lambda^2 + \lambda - 2 &= 0 \\ \lambda_{1,2} &= -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} \\ &= -\frac{1}{2} \pm \frac{3}{2} = 1 \text{ and } -2 \end{aligned}$$

   This implies $y_1 = e^x$ and $y_2 = e^{-2x}$, which means that the general solution is

$$y = C_1 e^x + C_2 e^{-2x}$$

2. **Two complex conjugate roots** $\lambda_{1,2} = p \pm i\,q$.

   This implies that

$$\begin{aligned}\tilde{y}_1 &= e^{px}[\cos(qx) + i\sin(qx)] \\ \tilde{y}_2 &= e^{px}[\cos(qx) - i\sin(qx)]\end{aligned}$$

   since $e^{iA} = \cos A + i \sin A$.

   But at this point we are interested in real solutions *only*. But we can take the following linear combination of the above functions:

$$\begin{aligned}y_1 &\equiv \frac{\tilde{y}_1 + \tilde{y}_2}{2} = e^{px}\cos(qx) \\ y_2 &\equiv \frac{\tilde{y}_1 - \tilde{y}_2}{2i} = e^{px}\sin(qx)\end{aligned}$$

   and have a new, *equivalent*, basis set. The new functions are both *real*, thus the general solution can be written as

$$y = e^{px}[C_1\cos(qx) + C_2\sin(qx)]$$

   One can easily verify that both y_1 and y_2 do (individually) meet the original equation.

   EXAMPLE:

$$\begin{aligned}y'' - 2y' + 10y &= 0 \\ \lambda_{1,2} &= 1 \pm \sqrt{1-10} \\ &= 1 \pm 3i\end{aligned}$$

   Thus

$$y = e^{x}[C_1\cos(3x) + C_2\sin(3x)]$$

   is the general solution.

3. **One (double) real root**.

   This can happen only when the original equation has the form of:

$$y'' + ay' + \frac{a^2}{4}y = 0$$

(i.e. $b = \frac{a^2}{4}$). Solving for λ, one gets:

$$\lambda_{1,2} = -\frac{a}{2} \pm 0$$

(double root). This gives us only one basic solution, namely

$$y_1 = e^{-\frac{a}{2}x}$$

We can find the other by the V-of-P technique. Let us substitute the following trial solution

$$c(x) \cdot e^{-\frac{a}{2}x}$$

into the equation getting (after we divide by $e^{-\frac{a}{2}x}$):

$$\begin{aligned} c'' - ac' + ac' &= 0 \\ c'' &= 0 \\ c(x) &= C_1 x + C_2 \end{aligned}$$

The trial solution thus becomes: $C_1 x e^{-\frac{a}{2}x} + C_2 e^{-\frac{a}{2}x}$, which clearly identifies

$$y_2 = x e^{-\frac{a}{2}x}$$

as the second basic solution.

Remember: For duplicate roots, the second solution can be obtained by multiplying the first basic solution by x.

EXAMPLE:

$$\begin{aligned} y'' + 8y' + 16y &= 0 \\ \lambda_{1,2} &= -4 \end{aligned}$$

The general solution is thus

$$y = e^{-4x}(C_1 + C_2 x)$$

Let's try finishing this as an initial-value problem [lest we forget]:

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= -3 \end{aligned}$$

This implies

$$C_1 = 1$$

and

$$\begin{aligned} -4C_1 + C_2 &= -3 \\ C_2 &= 1 \end{aligned}$$

The final answer:

$$y = (1+x)e^{-4x}$$

For a second-order equation, these three possibilities cover the whole story.

Non-homogeneous Case

When any such equation has a nonzero right hand side $r(x)$, there are two possible ways of building a particular solution y_p:

▷ Using the **variation-of-parameters** formulas derived earlier for the general case.

EXAMPLES:

1.

$$\begin{aligned} y'' + y &= \tan x \\ \lambda^2 + 1 &= 0 \\ \lambda_{1,2} &= \pm i \end{aligned}$$

implying that $\sin x$ and $\cos x$ are the two basic solutions of the homogeneous version. The old formulas give:

$$\begin{aligned} u' &= \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \sin x \\ u(x) &= -\cos x + C_1 \end{aligned}$$

and

$$\begin{aligned}
v' &= \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} \\
&= -\frac{\sin^2 x}{\cos x} = \cos x - \frac{1}{\cos x} \\
v(x) &= \sin x - \ln\left(\frac{1+\sin x}{\cos x}\right) + C_2
\end{aligned}$$

The final solution is thus

$$\begin{aligned}
y &= \{-\cos x \sin x + \sin x \cos x\} \\
&\quad - \cos x \ln\left(\frac{1+\sin x}{\cos x}\right) + C_1 \sin x + C_2 \cos x
\end{aligned}$$

The terms inside the curly brackets cancelling out, which happens frequently in these cases.

2.

$$y'' - 4y' + 4y = \frac{e^{2x}}{x}$$

Since $\lambda_{1,2} = 2 \pm 0$ [double root], the basic solutions are e^{2x} and xe^{2x}.

$$\begin{aligned}
u' &= \frac{\begin{vmatrix} 0 & xe^{2x} \\ \frac{e^{2x}}{x} & (1+2x)e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix}} = -1 \\
u(x) &= -x + C_1
\end{aligned}$$

and

$$\begin{aligned}
v' &= \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix}} = \frac{1}{x} \\
v(x) &= \ln x + C_2
\end{aligned}$$

Answer:

$$\begin{aligned} y &= C_1 e^{2x} + \tilde{C}_2 x e^{2x} - x e^{2x} + \ln x \cdot x e^{2x} \\ &= e^{2x}(C_1 + C_2 x + x \ln x) \end{aligned}$$

▷ **Special** cases of $r(x)$

- $r(x)$ is a **polynomial** in x:

Use a polynomial of the *same degree* but with undetermined coefficients as a trial solution for y_p.

EXAMPLE:

$$\begin{aligned} y'' + 2y' - 3y &= x \\ \lambda_{1,2} &= 1 \text{ and } -3 \\ y_1 &= e^x \text{ and } y_2 = e^{-3x} \end{aligned}$$

$$y_p = Ax + B$$

where A and B are found by substituting this y_p into the full equation and getting:

$$\begin{aligned} 2A - 3Ax - 3B &= x \\ A &= -\frac{1}{3} \\ B &= -\frac{2}{9} \end{aligned}$$

Answer:

$$y = C_1 e^x + C_2 e^{-3x} - \frac{x}{3} - \frac{2}{9}$$

Exceptional case: When $\lambda = 0$, this will not work unless the trial solution y_p is further multiplied by x (when $\lambda = 0$ is a *multiple* root, x has to be raised to the multiplicity of λ).

EXAMPLE:

$$\begin{aligned} y'' - 2y' &= x^2 + 1 \\ \lambda_{1,2} &= 0 \text{ and } 2 \\ y_1 &= 1 \text{ and } y_2 = e^{2x} \end{aligned}$$

$$y_p = Ax^3 + Bx^2 + Cx$$

where A, B and C are found by substituting:

$$\begin{aligned} 6Ax + 2B - 2(3Ax^2 + 2Bx + C) &= x^2 + 1 \\ A &= -\frac{1}{6} \\ B &= -\frac{1}{4} \\ C &= -\frac{3}{4}. \end{aligned}$$

Answer:

$$y = C_1 + C_2 e^{2x} - \frac{x^3}{6} - \frac{x^2}{4} - \frac{3x}{4}$$

- '**Exponential'** case:

$$r(x) \equiv k e^{\alpha x}$$

The trial solution is

$$y_p = A e^{\alpha x}$$

[with only A to be found; this is the undetermined coefficient of this case].

EXAMPLE:

$$\begin{aligned} y'' + 2y' + 3y &= 3e^{-2x} \\ \lambda_{1,2} &= -1 \pm \sqrt{2}\, i \end{aligned}$$

$$y_p = A e^{-2x}$$

substituted gives:

$$\begin{aligned} A(4 - 4 + 3)e^{-2x} &= 3e^{-2x} \\ A &= 1 \end{aligned}$$

$$y = e^{-x}[C_1 \sin(\sqrt{2}x) + C_2 \cos(\sqrt{2}x)] + e^{-2x}$$

Exceptional case: When $\alpha = \lambda$ [any of the roots], the trial solution must be first multiplied by x (to the power of the multiplicity of this λ).

EXAMPLE:

$$\begin{aligned} y'' + y' - 2y &= 3e^{-2x} \\ \lambda_{1,2} &= 1 \text{ and } -2 \end{aligned}$$

[same as α]!

$$y_p = Axe^{-2x}$$

substituted:

$$\begin{aligned} A(4x-4) + A(1-2x) - 2Ax &= 3 \\ A &= -1 \end{aligned}$$

[this follows from the absolute term, the x-proportional terms cancel out, as they must].

Answer:

$$y = C_1 e^x + (C_2 - x)e^{-2x}$$

- '**complex**' case:

$$r(x) = k_s e^{px} \sin(qx)$$

or

$$k_c e^{px} \cos(qx)$$

or a **combination** (sum) of both:

The trial solution is

$$[A \sin(qx) + B \cos(qx)]e^{px}$$

EXAMPLE:

$$\begin{aligned} y'' + y' - 2y &= 2e^{-x} \sin(4x) \\ \lambda_{1,2} &= 1 \text{ and } -2 \end{aligned}$$

as before.

$$y_p = [A \sin(4x) + B \cos(4x)]e^{-x}$$

substituted into the equation:

$$\begin{aligned}
&-18[A\sin(4x)+B\cos(4x)]\\
&-4[A\cos(4x)-B\sin(4x)] = 2\sin(4x)\\
&-18A+4B=2\\
&-4A-18B=0
\end{aligned}$$

solving, in matrix notation

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -18 & 4 \\ -4 & -18 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{9}{85} \\ \frac{2}{85} \end{bmatrix}$$

implying

$$y = C_1 e^x + C_2 e^{-2x} + \left(\frac{2}{85}\cos(4x) - \frac{9}{85}\sin(4x) \right) e^{-2x}$$

Exceptional case: When $\lambda = p + i\,q$ [*both* the real and purely imaginary parts must agree], the trial solution acquires the standard factor of x.

'Special-case' **summary:**

We would like to mention that all these special case can be covered by *one* and the same rule: When

$$r(x) = P_n(x)e^{\beta x}$$

where $P_n(x)$ is an n-degree polynomial in x, the trial solution is

$$Q_n(x)e^{\beta x}$$

where $Q_n(x)$ is also an n-degree polynomial, but with 'undetermined' (i.e. yet to be found) coefficients.

And the same exception: When β coincides with a root of the characteristic polynomial (of multiplicity ℓ) the trial solution must be further multiplied by x^ℓ.

If we allowed complex solutions, these rules would have covered it all. Since we don't, we have to spell it out differently for $\beta = p + i\,q$:

When

$$r(x) = [P_s(x)\sin(qx) + P_c(x)\cos(qx)]e^{px}$$

where $P_{s,c}$ are two polynomials of degree not higher than n [i.e. n is the higher of the two; also: one P may be identically equal to zero], the trial solution is:

$$[Q_s(x)\sin(qx) + Q_c(x)\cos(qx)]e^{px}$$

with *both* $Q_{s,c}$ being polynomials of degree n [no compromise here – they both have to be there, with the full degree, even if one P is missing].

Exception: If $p + i\,q$ coincides with one of the λs, the trial solution must be further multiplied by x raised to the λ's multiplicity [note that the conjugate root $p - i\,q$ will have the same multiplicity; use the multiplicity of *one* of these – don't double it].

Finally, if the right hand side is a *linear combination* (sum) of such terms, we use the superposition principle to construct the overall y_p. This means we find y_p individually for each of the distinct terms of $r(x)$, then add them together to build the final solution.

EXAMPLE:

$$\begin{aligned} y'' + 2y' - 3y &= x + e^{-x} \\ \lambda_{1,2} &= 1, -3 \end{aligned}$$

We break the right hand side into

$$r_1 \equiv x$$

and

$$r_2 \equiv e^{-x}$$

construct

$$y_{1p} = Ax + B$$

substituted into the equation with *only* x on the right hand side:

$$\begin{aligned} 2A - 3Ax - 3B &= x \\ A &= -\frac{1}{3} \\ B &= -\frac{2}{9} \end{aligned}$$

and then

$$y_{2p} = Ce^{-x}$$

substituted into the equation with r_2 only:

$$\begin{aligned} C - 2C - 3C &= 1 \\ C &= -\frac{1}{4} \end{aligned}$$

Answer:

$$y = C_1 e^x + C_2 e^{-3x} - \frac{x}{3} - \frac{2}{9} - \frac{1}{4} e^{-x}$$

Cauchy equation

looks like this:

$$(x - x_0)^2 y'' + a(x - x_0) y' + by = r(x)$$

where a, b and x_0 are specific constants.

x_0 is usually equal to 0, e.g.

$$x^2 y'' + 2xy' - 3y = x^5$$

There are two ways of solving it:

Converting

it to the previous case of a constant-coefficient equation (convenient when $r(x)$ is a polynomial in either x or $\ln x$ – try to figure our why). This conversion is achieved by introducing a new *independent* variable

$$t = \ln(x - x_0)$$

We have already derived the following set of formulas for performing such a conversion:

$$y' = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dot{y}}{x - x_0}$$

and

$$\begin{aligned} y'' &= \frac{d^2y}{dt^2} \cdot \left(\frac{dt}{dx} \right)^2 + \frac{dy}{dt} \cdot \frac{d^2t}{dx^2} \\ &= \frac{\ddot{y}}{(x - x_0)^2} - \frac{\dot{y}}{(x - x_0)^2} \end{aligned}$$

The original Cauchy equation thus becomes

$$\ddot{y} + (a-1)\dot{y} + by = r(x_0 + e^t)$$

which we solve by the old technique.

EXAMPLES:

$$\begin{aligned} x^2 y'' - 4xy' + 6y &= \frac{42}{x^4} \\ \ddot{y} - 5\dot{y} + 6y &= 42e^{-4t} \end{aligned}$$

$\Rightarrow \lambda_{1,2} = 2, 3$ and

$$y_p = Ae^{-4t}$$

substituted:

$$\begin{aligned} 16A + 20A + 6A &= 42 \\ A &= 1 \end{aligned}$$

Answer:

$$\begin{aligned} y &= C_1 e^{3t} + C_2 e^{2t} + e^{-4t} \\ &= C_1 x^3 + C_2 x^2 + \frac{1}{x^4} \end{aligned}$$

since $x = e^t$.

Another example:

$$y'' - \frac{y'}{x} - \frac{3y}{x^2} = \ln x + 1$$

(must be multiplied by x^2 first)

$$\ddot{y} - 2\dot{y} - 3y = te^{2t} + e^{2t}$$

$\lambda_{1,2} = 1 \pm \sqrt{1+3} = 3$ and -1,

$$\begin{aligned} y_p &= (At + B)e^{2t} \\ \dot{y}_p &= (2At + 2B + A)e^{2t} \\ \ddot{y}_p &= (4At + 4B + 4A)e^{2t} \end{aligned}$$

implying

$$\begin{aligned} 4A - 4A - 3A &= 1 \\ 4B + 4A - 4B - 2A - 3B &= 1 \end{aligned}$$

which means that $A = -\frac{1}{3}$ and $B = -\frac{5}{9}$.

Answer:

$$\begin{aligned} y &= C_1 e^{3t} + C_2 e^{-t} - \left(\frac{5}{9} + \frac{t}{3}\right) e^{2t} \\ &= C_1 x^3 + \frac{C_2}{x} - \left(\frac{5}{9} + \frac{\ln x}{3}\right) x^2 \end{aligned}$$

And yet another one:

$$x^2 y'' - 2xy' + 2y = 4x + \sin(\ln x)$$

$\Rightarrow$

$$\ddot{y} - 3\dot{y} + 2y = 4e^t + \sin(t)$$

$\Rightarrow \lambda_{1,2} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = 1,\ 2.$

$$\begin{aligned} y_{p_1} &= Ate^t \\ \dot{y}_{p_1} &= (At + A)e^t \\ \ddot{y}_{p_1} &= (At + 2A)e^t \end{aligned}$$

implying

$$\begin{aligned} 2A - 3A &= 4 \\ A &= -4 \end{aligned}$$

and

$$\begin{aligned} y_{p_2} &= B \sin t + C \cos t \\ \dot{y}_{p_2} &= B \cos t - C \sin t \\ \ddot{y}_{p_2} &= -B \sin t - C \cos t \end{aligned}$$

implying

$$\begin{aligned} -B + 3C + 2B &= 1 \\ -C - 3B + 2C &= 0 \end{aligned}$$

or

$$
\begin{aligned}
\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} B \\ C \end{bmatrix} &= \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \div 10 \\
&= \begin{bmatrix} \frac{1}{10} \\ \frac{3}{10} \end{bmatrix}
\end{aligned}
$$

1. Answer

$$
\begin{aligned}
y &= C_1 e^t + C_2 e^{2t} - 4te^t + \frac{1}{10}\sin(t) + \frac{3}{10}\cos(t) \\
&= C_1 x + C_2 x^2 - 4x \ln x + \frac{1}{10}\sin(\ln x) + \frac{3}{10}\cos(\ln x)
\end{aligned}
$$

Warning: To use the undetermined-coefficients technique (via the t transformation) the equation must have (or must be brought to) the form of:

$$x^2 y'' + axy' + by = r(x)$$

to use the V-of-P technique (which we are going to do now), the equation must have the form of

$$y'' + \frac{a}{x}y' + \frac{b}{x^2}y = r(x)$$

Direct technique:

more convenient when $r(x)$ is either *zero*, or does *NOT* have the *special* form mentioned above.

We substitute a trial solution $(x - x_0)^m$, with m yet to be determined, into the homogeneous Cauchy equation, and divide by $(x - x_0)^m$. This results in:

$$m^2 + (a-1)m + b = 0$$

a characteristic polynomial for m.

With two distinct real roots, we get our two basic solutions right away; with a duplicate root, we need an extra factor of $\ln(x - x_0)$ to construct the second basic solution; with two complex roots, we must go back to the 'conversion' technique.

EXAMPLES:

1.

$$\begin{aligned} x^2y'' + xy' - y &= 0 \\ m^2 - 1 &= 0 \\ m_{1,2} &= \pm 1 \\ y &= C_1 x + C_2 \frac{1}{x} \end{aligned}$$

2.

$$\begin{aligned} x^2y'' + 3xy' + y &= 0 \\ m^2 + 2m + 1 &= 0 \\ m_{1,2} &= -1 \\ y &= \frac{C_1}{x} + \frac{C_2}{x} \ln x \end{aligned}$$

3.

$$3(2x-5)^2y'' - (2x-5)y' + 2y = 0$$

First we have to rewrite it in the standard form of:

$$\begin{aligned} (x - \frac{5}{2})^2 y'' - \frac{1}{6}(x - \frac{5}{2})y' + \frac{1}{6}y &= 0 \\ m^2 - \frac{7}{6}m + \frac{1}{6} &= 0 \\ m_{1,2} &= \frac{1}{6},\ 1 \end{aligned}$$

$$\begin{aligned} y &= \tilde{C}_1(x - \frac{5}{2}) + \tilde{C}_2(x - \frac{5}{2})^{1/6} \\ &= C_1(2x-5) + C_2(2x-5)^{1/6}. \end{aligned}$$

4.

$$x^2y'' - 4xy' + 4y = 0$$

with $y(1) = 4$ and $y'(1) = 13$. First

$$\begin{aligned} m^2 - 5m + 4 &= 0 \\ m_{1,2} &= 1,\ 4 \\ y &= C_1 x + C_2 x^4 \end{aligned}$$

Then $C_1 + C_2 = 4$ and $C_1 + 4C_2 = 13$ yield $C_2 = 3$ and $C_1 = 1$.

Answer:

$$y = x + 3x^4$$

5.

$$x^2y'' - 4xy' + 6y = x^4 \sin x$$

To solve the homogeneous version:

$$\begin{aligned} m^2 - 5m + 6 &= 0 \\ m_{1,2} &= 2,\ 3 \end{aligned}$$

implying

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= x^3 \end{aligned}$$

To use V-of-P formulas the equation *must be first rewritten* in the 'standard' form of

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x^2 \sin x$$

which yields

$$\begin{aligned} u' &= \frac{\begin{vmatrix} 0 & x^3 \\ x^2 \sin x & 3x^2 \end{vmatrix}}{\begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}} = -x \sin x \\ u(x) &= x \cos x - \sin x + C_1 \end{aligned}$$

and

$$\begin{aligned} v' &= \frac{\begin{vmatrix} x^2 & 0 \\ 2x & x^2 \sin x \end{vmatrix}}{\begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}} = \sin x \\ v(x) &= -\cos(x) + C_2 \end{aligned}$$

Solution:

$$y = (C_1 - \sin x)x^2 + C_2x^3$$

Third and Higher-Order Linear ODEs

First we extend the general linear-equation results to higher orders. Explicitly, we mention the third order only, but the extension to higher orders is quite obvious:

$$y''' + f(x)y'' + g(x)y' + h(x)y = r(x)$$

has the following general solution:

$$y = C_1 y_1 + C_2 y_2 + C_3 y_3 + y_p$$

where y_1, y_2 and y_3 are *three* basic (linearly independent) solutions of the homogeneous version. There is no general analytical technique for finding them. Should these be known (obtained by whatever other means), we can construct a particular solution (to the full equation) y_p by variation of parameters (this time we skip the details), getting:

$$u' = \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ r & y_2'' & y_3'' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}$$

with a similar formula for v' and for w' (we need three of them, one for each basic solution). The pattern of these formulas should be obvious.

The corresponding constant-coefficient equation can be solved easily by constructing its *characteristic polynomial* and finding its *roots*, in a manner which is a trivial extension of the second-degree case. The main difficulty here is finding roots of higher-degree polynomials.

Special cases

of *higher-degree* polynomials which we know how to solve:

1.
$$x^n = a$$

   by finding all (n distinct) *complex* values of $\sqrt[n]{a}$, i.e.

$$\sqrt[n]{a}\left(\cos\frac{2\pi k}{n} + i \sin\frac{2\pi k}{n}\right)$$

when $a > 0$ and

$$-\sqrt[n]{-a}\left(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}\right)$$

when $a < 0$, both with $k = 0, 1, 2, ..., n-1$.

Examples:

(i) $\sqrt[4]{16} = 2, -2, 2i$ and $-2i$ (ii) $\sqrt[3]{-8} = -2$ and $1 \pm \sqrt{3}i$.

2. When 0 is one of the roots, it's trivial to find it, with its multiplicity.

   Example: $x^4 + 2x^3 - 4x^2 = 0$ has obviously 0 as a double root. Dividing the equation by x^2 makes it into a quadratic equation which can be easily solved.

3. When **coefficients** of an equation **add up** to 0, 1 must be one of the roots. The left hand side is divisible by $(x-1)$ [synthetic division], which reduces its order.

   Example:

   $$x^3 - 2x^2 + 3x - 2 = 0$$

   thus leads to $(x^3-2x^2+3x-2) \div (x-1) = x^2-x+2$ [quadratic polynomial].

4. When **coefficients** of the **odd** powers of x and coefficients of the **even** powers of x add up to the same two answers, then -1 is one of the roots and $(x+1)$ can be factored out.

   Example:

   $$x^3 + 2x^2 + 3x + 2 = 0$$

   leads to $(x^3+2x^2+3x+2) \div (x+1) = x^2+x+2$ and a quadratic equation.

5. One can cut the degree of an equation in half when the equation has **even powers** of x only by introducing $z = x^2$.

   Example:

   $$x^4 - 3x^2 - 4 = 0$$

   thus reduces to

   $$z^2 - 3z - 4 = 0$$

   which has two roots $z_{1,2} = -1,\ 4$. The roots of the original equation thus are: $x_{1,2,3,4} = i,\ -i,\ 2,\ -2$.

6. All powers divisible by 3 (4, 5, etc.). Use the same trick.

   Example:
$$x^6 - 3x^3 - 4 = 0$$
   Introduce $z = x^3$, solve for $z_{1,2} = -1,\ 4$ [same as before]. Thus $x_{1,2,3} = \sqrt[3]{-1} = -1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$ and $x_{4,5,6} = \sqrt[3]{4},\ \sqrt[3]{4}\left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i\right)$.

7. When a **multiple root** is suspected (the question may indicate: 'there is a triple root'), the following will help: each differentiation of a polynomial reduces the multiplicity of its every root by one. This means, for example, that a triple root becomes a single root of the polynomial's *second* derivative.

   Example:
$$x^4 - 5x^3 + 6x^2 + 4x - 8 = 0$$
   given there is a triple root. Differentiating twice:
$$12x^2 - 30x + 12 = 0$$
   $\Rightarrow x_{1,2} = \frac{1}{2},\ 2$. These must be substituted back into the original equation. Since 2 meets the original equation ($\frac{1}{2}$ does not), it must be its *triple* root, which can be factored out, thus: $(x^4 - 5x^3 + 6x^2 + 4x - 8) \div (x-2)^3 = x + 1$. The last root is thus, trivially, equal to -1.

8. One an take a slightly more sophisticated approach when it comes to multiple roots. As was already mentioned: each differentiation of the polynomial reduces the multiplicity of every root by one, but may (and usually does) introduce a lot of extra 'phoney' roots. These can be eliminated by taking the greatest common divisor (GCD) of the polynomial and its derivative, by using Euclid's algorithm, which works as follows:

   To find the GCD of two polynomials p and q, we divide one into the other to find the *remainder* (residue) of this operation (we are allowed to multiply the result by a constant to make it a monic polynomial): $r_1 = \text{Res}(p \div q)$, then $r_2 = \text{Res}(q \div r_1)$, $r_3 = \text{Res}(r_1 \div r_2)$, ... until the remainder becomes zero. The GCD is the previous (last nonzero) r.

   Example:
$$p(x) = x^8 - 28x^7 + 337x^6 - 2274x^5 + 9396x^4 - 24312x^3 + 38432x^2 - 33920x + 12800$$
$$s(x) = GCD(p, p') = x^5 - 17x^4 + 112x^3 - 356x^2 + 544x - 320$$

$$u(x) = GCD(s, s') = x^2 - 6x + 8$$

The last (quadratic) polynomial can be solved (and thus factorized) easily: $u = (x-2)(x-4)$. Thus 2 and 4 (each) must be a double root of s and a triple root of p. Taking $s \div (x-2)^2(x-4)^2 = x - 5$ reveals that 5 is a single root of s and therefore a double root of p. Thus, we have found all *eight* roots of p: 2, 2, 2, 4, 4, 4, 5 and 5.

Constant-coefficient equations

Similarly to solving second-order equations of this kind, we

- find the roots of the characteristic polynomial,
- based on these, construct the basic solutions of the homogeneous equation,
- find y_p by either V-P or (more commonly) undetermined-coefficient technique (when r is one of the special types).

Since the Cauchy equation is effectively a linear equation in disguise, we know how to solve it (beyond the second order) as well.

EXAMPLES:

1.

$$\begin{aligned} y^{iv} + 4y'' + 4y &= 0 \\ \lambda^4 + 4\lambda^2 + 4 &= 0 \\ z(&= \lambda^2)_{1,2} = -2 \\ \lambda_{1,2,3,4} &= \pm\sqrt{2}\, i \\ y &= C_1 \sin(\sqrt{2}x) + C_2 \cos(\sqrt{2}x) + C_3 x \sin(\sqrt{2}x) + C_4 x \cos(\sqrt{2}x). \end{aligned}$$

2.

$$\begin{aligned} y''' + y'' + y' + y &= 0 \\ \lambda_1 &= -1 \\ \lambda^2 + 1 &= 0 \\ \lambda_{2,3} &= \pm\, i \\ y &= C_1 e^{-x} + C_2 \sin x + C_3 \cos(x). \end{aligned}$$

3.

$$\begin{aligned}
y^v - 3y^{iv} + 3y''' - y'' &= 0 \\
\lambda_{1,2} &= 0 \\
\lambda_3 &= 1 \\
\lambda^2 - 2\lambda + 1 &= 0 \\
\lambda_{4,5} &= 1 \\
y &= C_1 + C_2x + C_3e^x + C_4xe^x + C_5x^2e^x.
\end{aligned}$$

4.

$$\begin{aligned}
y^{iv} - 5y'' + 4y &= 0 \\
z(&= \lambda^2)_{1,2} = 1,\ 4 \\
\lambda_{1,2,3,4} &= \pm 2,\ \pm 1 \\
y &= C_1e^x + C_2e^{-x} + C_3e^{2x} + C_4e^{-2x}
\end{aligned}$$

5.

$$\begin{aligned}
y''' - y' &= 10\cos(2x) \\
\lambda_{1,2,3} &= 0,\ \pm 1 \\
y_p &= A\sin(2x) + B\cos(2x) \\
& \quad -10\cos(2x) + 10\sin(2x) = 10\cos(2x) \\
A &= -1, B = 0 \\
y &= C_1 + C_2e^x + C_3e^{-x} - \sin(2x).
\end{aligned}$$

6.

$$\begin{aligned}
y''' - 2y'' &= x^2 - 1 \\
\lambda_{1,2} &= 0,\ \lambda_3 = 2. \\
y_p &= Ax^4 + Bx^3 + Cx^2 \\
& \quad -24Ax^2 + 24Ax - 12Bx + 6B - 4C = x^2 - 1 \\
A &= -\frac{1}{24},\ B = -\frac{1}{12},\ C = \frac{1}{8} \\
y &= C_1 + C_2x + C_3e^{2x} - \frac{x^4}{24} - \frac{x^3}{12} + \frac{x^2}{8}.
\end{aligned}$$

Sets of Linear, First-Order, Constant-Coefficient ODEs

First we need to complete our review of

Matrix Algebra

Matrix inverse & determinant
There are two ways of finding these

▷ The **'classroom' algorithm**

which requires $\propto n!$ number of operations and becomes not just impractical, but virtually impossible to use (even for supercomputers) when n is large (>20). Yet, for small matrices ($n \le 4$) it's fine, and we actually prefer it. It works like this:

- In a 2×2 case, it's trivial: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$ where the denominator is the determinant.

  Example: $\begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{5}{22} & \frac{-4}{22} \\ \frac{3}{22} & \frac{2}{22} \end{bmatrix}$, 22 being the determinant.

- The 3×3 case is done in **four steps**.

  Example: $\begin{bmatrix} 2 & 4 & -1 \\ 0 & 3 & 2 \\ 2 & 1 & 4 \end{bmatrix}^{-1}$

1. Construct a 3×3 matrix of all 2×2 *subdeterminants* (striking out one row and one column – organize the answers accordingly):

$$\begin{bmatrix} 10 & -4 & -6 \\ 17 & 10 & -6 \\ 11 & 4 & 6 \end{bmatrix}$$

2. *Transpose* the answer:

10	17	11
-4	10	4
-6	-6	6

3. *Change* the *sign* of every other element (using the following checkerboard scheme:

$+$	$-$	$+$
$-$	$+$	$-$
$+$	$-$	$+$

thus:

10	-17	11
4	10	-4
-6	6	6

4. *Divide by* the *determinant*, which can be obtained easily be multiplying ('scalar' product) the first row of the original matrix by the first column of the last matrix (or vice versa $-$ one can also use the second or third row/column – column/row):

$\frac{10}{42}$	$\frac{-17}{42}$	$\frac{11}{42}$
$\frac{4}{42}$	$\frac{10}{42}$	$\frac{-4}{42}$
$\frac{-6}{42}$	$\frac{6}{42}$	$\frac{6}{42}$

- Essentially the same algorithm can be used for 4×4 matrices and beyond, but it becomes increasingly impractical and soon enough virtually impossible to carry out.

▷ The **'practical' algorithm**

requires $\propto n^3$ operations and can be easily converted into a computer code:

1. The original $(n \times n)$ matrix is *extended* to an $n \times 2n$ matrix by appending it with the $n \times n$ unit matrix.

2. By using one of the following **three** 'elementary' **operations** we make the original matrix into the *unit matrix*, while the appended part results in the desired inverse:

   1. A (full) row can be divided by any nonzero number [this is used to make the main-diagonal elements equal to 1, one by one].
   2. A multiple of a row can be added to (or subtracted from) any other row [this is used to make the non-diagonal elements of each column equal to 0].

3. Two rows can be interchanged whenever necessary [when a main-diagonal element is zero, interchange the row with any *subsequent* row which has a nonzero element in that position - if none exists the matrix is *singular*].

The product of the numbers we found on the main diagonal (and had to divide by), further multiplied by -1 if there has been an *odd* number of interchanges, is the matrix' **determinant**.

- A 4×4 EXAMPLE:

3	0	1	4	1	0	0	0	$\div 3$
1	-1	2	1	0	1	0	0	$-\frac{1}{3}r_1$
3	1	-1	1	0	0	1	0	$-r_1$
-2	0	-1	1	0	0	0	1	$+\frac{2}{3}r_1$

1	0	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	
0	-1	$\frac{5}{3}$	-$\frac{1}{3}$	-$\frac{1}{3}$	1	0	0	$\div(-1)$
0	1	-2	-3	-1	0	1	0	$+r_2$
0	0	-$\frac{1}{3}$	$\frac{11}{3}$	$\frac{2}{3}$	0	0	1	

1	0	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	0	0	0	$+r_3$
0	1	-$\frac{5}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	-1	0	0	$-5r_3$
0	0	-$\frac{1}{3}$	-$\frac{10}{3}$	-$\frac{4}{3}$	1	1	0	$\div(-\frac{1}{3})$
0	0	-$\frac{1}{3}$	$\frac{11}{3}$	$\frac{2}{3}$	0	0	1	$-r_3$

1	0	0	-2	-1	1	1	0	$+\frac{2}{7}r_4$
0	1	0	17	7	-6	-5	0	$-\frac{17}{7}r_4$
0	0	1	10	4	-3	-3	0	$-\frac{10}{7}r_4$
0	0	0	7	2	-1	-1	1	$\div 7$

1	0	0	0	$-\frac{3}{7}$	$\frac{5}{7}$	$\frac{5}{7}$	$\frac{2}{7}$
0	1	0	0	$\frac{15}{7}$	-$\frac{25}{7}$	-$\frac{18}{7}$	-$\frac{17}{7}$
0	0	1	0	$\frac{8}{7}$	-$\frac{11}{7}$	-$\frac{11}{7}$	-$\frac{10}{7}$
0	0	0	1	$\frac{2}{7}$	-$\frac{1}{7}$	-$\frac{1}{7}$	$\frac{1}{7}$

.

The last matrix is the inverse of the original matrix, as can be easily verified [no interchanges were needed]. The determinant is $3 \times (-1) \times (-\frac{1}{3}) \times 7 = 7$.

Solving n equations for m unknowns

For an $n \times n$ *non-singular* problems with n 'small' we can use the matrix inverse: $\mathbb{C}\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = \mathbb{C}^{-1}\mathbf{b}$, but this is not very practical beyond 2×2.

$\triangleright$ The fully **general technique**

applicable to singular as well as n by m problems works like this:

1. *Extend* $\mathbb{C}$ by an extra column $\mathbf{b}$.
2. Using 'elementary operations' make the original $\mathbb{C}$-part into the *unit matrix*. If you succeed, the $\mathbf{b}$-part of the matrix is the (unique) solution.

   This cannot work when the number of equations and the number of unknowns don't match. Furthermore, we may run into difficulty for the following two reasons:

   1. We may come to a column which has 0 on the main diagonal *and all elements below it*. This column will be then skipped (we will try to get 1 in the *same position* of the next column).
   2. Discarding the columns we skipped, we may end up with fewer columns than rows [resulting in some extra rows with only zeros in their $\mathbb{C}$-part], or the other way round [resulting in some (nonzero) extra columns, which we treat in the same manner as those columns which were skipped]. The final number of 1's [on the main diagonal] is the rank of $\mathbb{C}$.

   We will call the result of this part the matrix echelon form of the equations.
3. To *interpret* the answer we do this:

   1. If there are any 'extra' (zero $\mathbb{C}$-part) rows, check the *corresponding* $\mathbf{b}$ elements. If they are all equal to zero, we delete the extra (redundant) rows and go to the next step; if we find even a single non-zero element among them, the original system of equations is inconsistent, and there is *no solution*.
   2. Each of the 'skipped' columns represents an unknown whose value can be chosen arbitrarily. Each row then provides an expression for one of the remaining unknowns (in terms of the 'freely chosen' ones). Note that when there are no 'skipped' columns, the solution is just a point in m (number of unknowns) dimensions, one 'skipped' column results in a straight line, two 'skipped' columns in a plane, etc.

Since the first two steps of this procedure are quite straightforward, we give **EXAMPLES** of the interpretation part only:

1.

1	3	0	2	0	2
0	0	1	3	0	1
0	0	0	0	1	4
0	0	0	0	0	0

means that x_2 and x_4 are the 'free' parameters (often, they would be renamed c_1 and c_2, or A and B). The solution can thus be written as $\begin{cases} x_1 = 2 - 3x_2 - 2x_4 \\ x_3 = 1 - 3x_4 \\ x_5 = 4 \end{cases}$

or, in a vector-like manner:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} c_1 + \begin{bmatrix} -2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} c_2$$

Note that this represents a (unique) plane in a five-dimensional space; the 'point' itself and the two directions (coefficients of c_1 and c_2) can be specified in infinitely many different (but equivalent) ways.

2.

1	3	0	0	0	−2	5
0	0	1	0	0	3	2
0	0	0	1	0	1	0
0	0	0	0	1	4	3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} c_1 + \begin{bmatrix} 2 \\ 0 \\ -3 \\ -1 \\ -4 \\ 1 \end{bmatrix} c_2$$

Eigenvalues & Eigenvectors
of a **square** matrix.

If, for a square $(n \times n)$ matrix $\mathbb{A}$, we can find a *non-zero* [column] vector $\mathbf{x}$ and a (scalar) number λ such that

$$\mathbb{A}\mathbf{x} = \lambda\mathbf{x}$$

then λ is the matrix' eigenvalue and $\mathbf{x}$ is its **right eigenvector** (similarly $\mathbf{y}^T\mathbb{A} = \lambda\mathbf{y}^T$ would define its left eigenvector $\mathbf{y}^T$, this time a *row* vector). This means that we seek a non-zero solutions of

$$(\mathbb{A} - \lambda\mathbb{I})\,\mathbf{x} \equiv \mathbf{0}$$

which further implies that $\mathbb{A} - \lambda\mathbb{I}$ must be *singular*: $\det(\mathbb{A} - \lambda\mathbb{I}) = 0$.

The left hand side of the last equation is an n^{th}-degree polynomial in λ which has (counting multiplicity) n [possibly complex] roots. These roots are the **eigenvalues** of $\mathbb{A}$; one can easily see that for each distinct root one can find at least one right (and at least one left) eigenvector, by solving the above equation for $\mathbf{x}$ (λ being known now).

It is easy to verify that

$$\begin{aligned}
&\det(\lambda\mathbb{I} - \mathbb{A}) = \lambda^n - \lambda^{n-1} \cdot \mathrm{Tr}(\mathbb{A}) \\
&+\lambda^{n-2} \cdot \{\text{sum of all } 2 \times 2 \text{ major subdeterminants}\} \\
&-\lambda^{n-3} \cdot \{\text{sum of all } 3 \times 3 \text{ major subdeterminants}\} \\
&+ .... \pm \det(\mathbb{A})
\end{aligned}$$

where $\mathrm{Tr}(\mathbb{A})$ is the sum of all main-diagonal elements. This is called the **characteristic polynomial** of $\mathbb{A}$, and its roots are the only eigenvalues of $\mathbb{A}$.

EXAMPLES:

1.

2	3
1	−2

has $\lambda^2 - 0 \cdot \lambda - 7$ as its characteristic polynomial, which means that the eigenvalues are $\lambda_{1,2} = \pm\sqrt{7}$.

2.

3	−1	2
0	4	2
2	−1	3

$\lambda^3 - 10\lambda^2 + (12 + 14 + 5)\lambda - 22$. The coefficients add up to 0. This implies that $\lambda_1 = 1$ and [based on $\lambda^2 - 9\lambda + 22 = 0$] $\lambda_{2,3} = \frac{9}{2} \pm \frac{\sqrt{7}}{2} i$.

3. $\begin{array}{|c|c|c|c|} \hline 2 & 4 & -2 & 3 \\ \hline 3 & 6 & 1 & 4 \\ \hline -2 & 4 & 0 & 2 \\ \hline 8 & 1 & -2 & 4 \\ \hline \end{array}$

   $\lambda^4 - 12\lambda^3 + (0-4+4-4+20-16)\lambda^2 - (-64-15-28-22)\lambda + (-106) = 0$ [note there are $\binom{4}{2} = 6$ and $\binom{4}{3} = 4$ major subdeterminants of the 2×2 and 3×3 size, respectively] $\Rightarrow \lambda_1 = -3.2545$, $\lambda_2 = 0.88056$, $\lambda_3 = 3.3576$ and $\lambda_4 = 11.0163$ [these were obtained from our general formula for fourth-degree polynomials].

The corresponding (right) **eigenvectors** can be now found by solving the corresponding *homogenous* set of equations with a *singular* matrix of coefficients [therefore, there must be at least one *nonzero* solution – which, furthermore, can be multiplied by an arbitrary constant]. The number of linearly independent (LI) solutions cannot be bigger than the *multiplicity* of the corresponding eigenvalue; establishing their correct number is an important part of the answer.

EXAMPLES:

1. Using $\mathbb{A} = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & -2 \\ \hline \end{array}$ [one of our previous examples] $(\mathbb{A} - \lambda_1 \mathbb{I})\,\mathbf{x} = \mathbf{0}$ amounts to

   $$\begin{array}{|c|c||c|} \hline 2-\sqrt{7} & 3 & 0 \\ \hline 1 & -2-\sqrt{7} & 0 \\ \hline \end{array}$$

   with the second equation being a multiple of the first [check it!]. We thus have to solve only $x_1 - (2+\sqrt{7})x_2 = 0$, which has the following general solution: $\begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \end{array} = \begin{array}{|c|} \hline 2+\sqrt{7} \\ \hline 1 \\ \hline \end{array} c$, where c is arbitrary [geometrically, the solution represents a straight line in the x_1-x_2 plane, passing through the origin]. Any such vector, when pre-multiplied by $\mathbb{A}$, increases in length by a factor of $\sqrt{7}$, without changing direction (check it too). Similarly, replacing λ_1 by $\lambda_2 = -\sqrt{7}$, we would be getting $\begin{array}{|c|} \hline 2-\sqrt{7} \\ \hline 1 \\ \hline \end{array} c$ as the corresponding eigenvector.

   There are many equivalent ways of expressing it, $\begin{array}{|c|} \hline -3 \\ \hline 2+\sqrt{7} \\ \hline \end{array} \tilde{c}$ is one of them.

2. A **double eigenvalue** may possess either one or two linearly independent eigenvectors:

1. The unit 2×2 matrix has $\lambda = 1$ as its duplicate eigenvalue, $\boxed{\begin{array}{c} 1 \\ \hline 0 \end{array}}$ and $\boxed{\begin{array}{c} 0 \\ \hline 1 \end{array}}$ are two LI eigenvectors [the general solution to $\boxed{\begin{array}{cc||c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}}$]. This implies that *any* vector is an eigenvector of the unit matrix..

2. The matrix $\boxed{\begin{array}{c|c} 1 & 2 \\ \hline 0 & 1 \end{array}}$ has the same duplicate eigenvalue of $+1$ [in general, the main diagonal elements of an upper-triangular matrix are its eigenvalues], but solving $\boxed{\begin{array}{c|c||c} 0 & 2 & 0 \\ \hline 0 & 0 & 0 \end{array}}$ i.e. $2x_2 = 0$ has only *one* LI solution, namely $\boxed{\begin{array}{c} 1 \\ \hline 0 \end{array}}\, c$

Finding eigenvectors and eigenvalues of a matrix represents the main step in solving **sets of ODE**s; we will present our further examples in that context. So let us now return to these:

Set (system) of differential equations

of *first order, linear,* and with *constant coefficients* typically looks like this:

$$\begin{aligned} y_1' &= 3y_1 + 4y_2 \\ y_2' &= 3y_1 - y_2 \end{aligned}$$

[the example is of the *homogeneous* type, as each term is either y_i or y_i' proportional]. The same set can be conveniently expressed in **matrix notation** as

$$\mathbf{y}' = \mathbb{A}\mathbf{y}$$

where $\mathbb{A} = \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ [both y_1 and y_2 are function of x].

The Main Technique

for constructing a solution to any such set of n DEs is very similar to what we have seen in the case of one (linear, constant-coefficient, homogeneous) DE, namely: We first try to find n *linearly independent* basic solutions (all having the $\boxed{\begin{array}{c} y_1 \\ \hline y_2 \\ \hline \vdots \\ \hline y_n \end{array}}$ form), then build the general solution as a linear combination (with arbitrary coefficients) of these.

It happens that the **basic solutions** can be constructed with the help of matrix algebra. To find them, we use the following *trial* solution:

$$\mathbf{y}_T = \mathbf{q} \cdot e^{\lambda x}$$

where $\mathbf{q}$ is a constant (n-dimensional) vector. Substituting into $\mathbf{y}' = \mathbb{A}\mathbf{y}$ and cancelling the (*scalar*) $e^{\lambda x}$ gives: $\lambda \mathbf{q} = \mathbb{A}\mathbf{q}$, which means λ can be any one of the *eigenvalues* of $\mathbb{A}$ and $\mathbf{q}$ be the corresponding *eigenvector*. If we find n of these (which is the case with *simple* eigenvalues) the job is done; we have effectively constructed a general solution to our set of DEs.

EXAMPLES:

1. Solve $\mathbf{y}' = \mathbb{A}\mathbf{y}$, where $\mathbb{A} = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 3 & -1 \\ \hline\end{array}$. The characteristic equation is: $\lambda^2 - 2\lambda - 15 = 0 \Rightarrow \lambda_{1,2} = 1 \pm \sqrt{16} = -3$ and 5. The corresponding eigenvectors (we will call them $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$) are the solutions to $\begin{array}{|c|c||c|}\hline 6 & 4 & 0 \\ \hline 3 & 2 & 0 \\ \hline\end{array} \Rightarrow 3q_1^{(1)} + 2q_2^{(1)} = 0 \Rightarrow \mathbf{q}^{(1)} = \begin{array}{|c|}\hline 2 \\ \hline -3 \\ \hline\end{array}\, c_1$, and $\begin{array}{|c|c||}\hline -2 & 4 \\ \hline 3 & -6 \\ \hline\end{array}$ [from now on we will *assume* a zero right hand side]$\Rightarrow q_1^{(2)} - 2q_2^{(2)} = 0 \Rightarrow \mathbf{q}^{(2)} = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}\, c_2$.

   The final, general solution is thus

   $$\mathbf{y} = c_1 \begin{array}{|c|}\hline 2 \\ \hline -3 \\ \hline\end{array} e^{-3x} + c_2 \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} e^{5x}$$

   Or, if you prefer, more explicitly: $\begin{array}{c} y_1 = 2c_1 e^{-3x} + 2c_2 e^{5x} \\ y_2 = -3c_1 e^{-3x} + c_2 e^{5x} \end{array}$ where c_1 and c_2 can be chosen arbitrarily.

   Often, they are specified via initial conditions, e.g. $y_1(0) = 2$ and $y_2(0) = -3$ $\Rightarrow \begin{array}{c} 2c_1 + 2c_2 = 2 \\ c_1 - 3c_2 = -3 \end{array} \Rightarrow c_1 = 1$ and $c_2 = 0 \Rightarrow \begin{array}{c} y_1 = 2e^{-3x} \\ y_2 = -3e^{-3x} \end{array}$.

2. Let us now tackle a three-dimensional problem, with $\mathbb{A} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

The characteristic equation is $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0 \Rightarrow \lambda_1 = -1$ and the roots of $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_{2,3} = 1$ and 2. The respective eigenvectors are: $\left[\begin{array}{ccc|} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|} 1 & 0 & \frac{1}{5} \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{q}^{(1)} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} c_1$, $\left[\begin{array}{ccc|} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{array}\right] \Rightarrow \mathbf{q}^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} c_2$, and $\left[\begin{array}{ccc|} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{q}^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} c_3$. One can easily verify the correctness of each eigenvector by a simple multiplication, e.g. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

The general solution is thus

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} e^{-x} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{x} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2x}$$

The case of **Double (Multiple) Eigenvalue**
For each such eigenvalue we must first find *all* possible solutions of the

$$\mathbf{q}e^{\lambda x}$$

type (i.e. find all LI eigenvectors), then (if we get fewer eigenvectors than the multiplicity of λ) we have to find all possible solutions having the form of

$$(\mathbf{q}x + \mathbf{s})e^{\lambda x}$$

where $\mathbf{q}$ and $\mathbf{s}$ are two constant vectors to be found by substituting this (trial) solution into the basic equation $\mathbf{y}' = \mathbb{A}\mathbf{y}$. As a result we get

$$\mathbf{q} = (\mathbb{A} - \lambda\mathbb{I})\mathbf{q}\,x + (\mathbb{A} - \lambda\mathbb{I})\mathbf{s}$$

This further implies that $\mathbf{s}$ must be a solution to

$$(\mathbb{A} - \lambda\mathbb{I})^2\mathbf{s} = \mathbf{0}$$

And, if still not done, we have to proceed to

$$(\mathbf{q}\frac{x^2}{2!} + \mathbf{s}x + \mathbf{u})e^{\lambda x}$$

which implies

$$\begin{aligned}(\mathbb{A} - \lambda\mathbb{I})\mathbf{q} &= \mathbf{0} \\ \mathbf{q} &= (\mathbb{A} - \lambda\mathbb{I})\mathbf{s} \\ \mathbf{s} &= (\mathbb{A} - \lambda\mathbb{I})\mathbf{u}\end{aligned}$$

where, clearly, $\mathbf{u}$ is a solution to

$$(\mathbb{A} - \lambda\mathbb{I})^3\mathbf{u} = \mathbf{0}$$

etc.

EXAMPLES:

1.

$$\mathbf{A} = \begin{array}{|c|c|c|}\hline 5 & 2 & 2 \\ \hline 2 & 2 & \text{-}4 \\ \hline 2 & \text{-}4 & 2 \\ \hline\end{array}$$

has $\lambda^3 - 9\lambda^2 + 108$ as its characteristic polynomial [hint: there is a double root] $\Rightarrow$ $3\lambda^2 - 18\lambda = 0$ has two roots, 0 [does not check] and 6 [checks]. Furthermore, $(\lambda^3 - 9\lambda^2 + 108) \div (\lambda - 6)^2 = \lambda + 3 \Rightarrow$ the three eigenvalues are -3 and 6 [duplicate].

Using $\lambda = -3$ we get:

$$\begin{array}{|c|c|c||}\hline 8 & 2 & 2 \\ \hline 2 & 5 & -4 \\ \hline 2 & -4 & 5 \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 1 & 0 & \frac{1}{2} \\ \hline 0 & 1 & -1 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \Rightarrow c_1 \begin{array}{|c|}\hline -1 \\ \hline 2 \\ \hline 2 \\ \hline\end{array}$$

which, when multiplied by e^{-3x}, gives the first basic solution.

Using $\lambda = 6$ yields:

$$\begin{array}{|c|c|c||}\hline -1 & 2 & 2 \\ \hline 2 & -4 & -4 \\ \hline 2 & -4 & -4 \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 1 & -2 & -2 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \Rightarrow c_2 \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 0 \\ \hline\end{array} + c_3 \begin{array}{|c|}\hline 2 \\ \hline 0 \\ \hline 1 \\ \hline\end{array}$$

which, when multiplied by e^{6x}, supplies the remaining two basic solutions.

2.

$$\mathbb{A} = \begin{array}{|c|c|c|}\hline 1 & -3 & 1 \\ \hline 2 & -1 & -2 \\ \hline 2 & -3 & 0 \\ \hline\end{array}$$

has $\lambda^3 - 3\lambda - 2$ as its characteristic polynomial, with roots: $\lambda_1 = -1$ [one of our rules] $\Rightarrow (\lambda^3 - 3\lambda - 2) \div (\lambda + 1) = \lambda^2 - \lambda - 2 \Rightarrow \lambda_2 = -1$ and $\lambda_3 = 2$. So again, there is one duplicate root.

For $\lambda = 2$ we get:

$$\begin{array}{|c|c|c||}\hline -1 & -3 & 1 \\ \hline 2 & -3 & -2 \\ \hline 2 & -3 & -2 \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 1 & 0 & -1 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \Rightarrow c_1 \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline 1 \\ \hline\end{array} e^{2x}$$

For $\lambda = -1$ we get:

$$\begin{array}{|c|c|c||}\hline 2 & -3 & 1 \\ \hline 2 & 0 & -2 \\ \hline 2 & -3 & 1 \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 1 & 0 & -1 \\ \hline 0 & 1 & -1 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \Rightarrow c_2 \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline\end{array} e^{-x}$$

[a *single* solution only]. The challenge is to construct the other (last) solution. Squaring the previous matrix yields

$$\begin{array}{|c|c|c||}\hline 0 & -9 & 9 \\ \hline 0 & 0 & 0 \\ \hline 0 & -9 & 9 \\ \hline\end{array} \Rightarrow \begin{array}{|c|c|c||}\hline 0 & 1 & -1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array}$$

whose general solution is a linear combination of

$$\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline 0 \\ \hline\end{array} \text{ and } \begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline 1 \\ \hline\end{array}$$

(either of these could be taken as **s** - we will take the first one): Since

$$\begin{array}{|c|c|c||}\hline 2 & -3 & 1 \\ \hline 2 & 0 & -2 \\ \hline 2 & -3 & 1 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline 0 \\ \hline\end{array} = \begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline 2 \\ \hline\end{array}$$

the third basic solution is:

$$c_3 \left(\begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline 2 \\ \hline\end{array} x + \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline 0 \\ \hline\end{array} \right) e^{-x} \equiv c_3 \begin{array}{|c|}\hline 2x+1 \\ \hline 2x \\ \hline 2x \\ \hline\end{array} e^{-x}$$

3.

$$\mathbb{A} = \begin{bmatrix} 42 & -9 & 9 \\ -12 & 39 & -9 \\ -28 & 21 & 9 \end{bmatrix}$$

$\Rightarrow \lambda^3 - 90\lambda^2 + 2700\lambda - 27000$ [hint: triple root] $\Rightarrow 6\lambda - 180 = 0$ has a single root of 30 [$\Rightarrow$ triple root of the original polynomial]. Finding eigenvectors:

$$\begin{bmatrix} 12 & -9 & 9 \\ -12 & 9 & -9 \\ -28 & 21 & -21 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \text{ and } c_2 \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$$

are the corresponding eigenvectors [only two] which, when multiplied by e^{30x} yield the first two basic solutions.

Squaring the previous matrix yields the zero matrix (any vector is a solution - we just have to be careful not to take a linear combination of the previous two). We thus take

$$\mathbf{s} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which implies

$$\mathbf{q} = \begin{bmatrix} 12 & -9 & 9 \\ -12 & 9 & -9 \\ -28 & 21 & -21 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -21 \end{bmatrix}$$

The third basic solution is thus $c_3 \left[\begin{bmatrix} 9 \\ -9 \\ -21 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] e^{30x} = c_3 \begin{bmatrix} 9x \\ -9x \\ -21x + 1 \end{bmatrix} e^{30x}$.

4.

$$\mathbb{A} = \begin{bmatrix} -103 & -53 & 41 \\ 160 & 85 & -100 \\ 156 & 131 & -147 \end{bmatrix}$$

$\Rightarrow \lambda^3 + 165\lambda^2 + 9075\lambda + 166375$ [hint: triple root] $\Rightarrow 6\lambda + 330 = 0 \Rightarrow \lambda = -55$ [checks] $\Rightarrow$

$$\begin{bmatrix} -48 & -53 & 41 \\ 160 & 140 & -100 \\ 156 & 131 & -92 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix}$$

is the only eigenvector (this, multiplied by e^{-55x}, provides the first basic solution). Squaring the matrix, we get

$$\left.\begin{array}{|r|r|r|}\hline 220 & 495 & -440 \\ \hline -880 & -1980 & 1760 \\ \hline -880 & -1980 & 1760 \\ \hline \end{array}\right\| \Rightarrow \left.\begin{array}{|r|r|r|}\hline 1 & \frac{4}{9} & -2 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}\right\|$$

A possible solution for **s** is thus

$$\begin{array}{|c|}\hline 2 \\ \hline 0 \\ \hline 1 \\ \hline \end{array}$$

which implies that

$$\mathbf{q} = \left.\begin{array}{|r|r|r|}\hline -48 & -53 & 41 \\ \hline 160 & 140 & -100 \\ \hline 156 & 131 & -92 \\ \hline \end{array}\right\| \cdot \begin{array}{|c|}\hline 2 \\ \hline 0 \\ \hline 1 \\ \hline \end{array} = \begin{array}{|r|}\hline -55 \\ \hline 220 \\ \hline 220 \\ \hline \end{array}$$

The second basic solution is thus

$$c_2(\mathbf{q}x + \mathbf{s})e^{-55x} = c_2 \begin{array}{|c|}\hline -55x + 2 \\ \hline 220x \\ \hline 220x + 1 \\ \hline \end{array} e^{-55x}$$

Finally, cubing $(\mathbb{A} + 55\mathbb{I})$ results in a zero matrix, so we can choose *any* **u**, as long as it's not a linear combination of the previous two vectors, e.g.

$$\begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline 0 \\ \hline \end{array}$$

This implies that

$$\mathbf{s} = \left.\begin{array}{|r|r|r|}\hline -48 & -53 & 41 \\ \hline 160 & 140 & -100 \\ \hline 156 & 131 & -92 \\ \hline \end{array}\right\| \cdot \begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline 0 \\ \hline \end{array} = \begin{array}{|r|}\hline -53 \\ \hline 140 \\ \hline 131 \\ \hline \end{array}$$

and

$$\mathbf{q} = \left.\begin{array}{|r|r|r|}\hline -48 & -53 & 41 \\ \hline 160 & 140 & -100 \\ \hline 156 & 131 & -92 \\ \hline \end{array}\right\| \cdot \begin{array}{|r|}\hline -53 \\ \hline 140 \\ \hline 131 \\ \hline \end{array} = \begin{array}{|r|}\hline 495 \\ \hline -1980 \\ \hline -1980 \\ \hline \end{array}$$

The last basic solution thus equals

$$c_3\left(\mathbf{q}\frac{x^2}{2}+\mathbf{s}x+\mathbf{u}\right)e^{-55x} = c_3\begin{bmatrix}\frac{495}{2}x^2-53x\\ -990x^2+140x+1\\ -990x^2+131x\end{bmatrix}e^{-55x}$$

Complex Eigenvalues/Vectors

we first write the corresponding solution in a complex form, using the regular procedure. We then replace each conjugate pair of basic solutions by the real and imaginary part (of either solution).

EXAMPLE:

$$\mathbf{y}' = \begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}\mathbf{y}$$

$\Rightarrow \lambda^2-6\lambda+11 \Rightarrow \lambda_{1,2} = 3\pm\sqrt{2}\,i \Rightarrow$

$$\left[\begin{array}{cc|} -1-\sqrt{2}\,i & -1\\ 3 & 1-\sqrt{2}\,i\end{array}\right] \Rightarrow \begin{bmatrix}1-\sqrt{2}\,i\\ -3\end{bmatrix}$$

is the eigenvector corresponding to $\lambda_1 = 3+\sqrt{2}\,i$ [its complex conjugate corresponds to $\lambda_2 = 3-\sqrt{2}\,i$]. This means that the two basic solutions (in their complex form) are

$$\begin{bmatrix}1-\sqrt{2}\,i\\ -3\end{bmatrix}e^{(3+\sqrt{2}\,i)x} \equiv$$

$$\begin{bmatrix}1-\sqrt{2}\,i\\ -3\end{bmatrix}\left[\cos(\sqrt{2}x)+i\sin(\sqrt{2}x)\right]e^{3x}$$

and its complex conjugate [$i\to -i$]. Equivalently, we can use the real and imaginary part of either of these [up to a sign, the same answer] to get:

$$\begin{aligned}\mathbf{y} \;=\; & c_1\begin{bmatrix}\cos(\sqrt{2}x)+\sqrt{2}\sin(\sqrt{2}x)\\ -3\cos(\sqrt{2}x)\end{bmatrix}e^{3x}\\ & +c_2\begin{bmatrix}-\sqrt{2}\cos(\sqrt{2}x)+\sin(\sqrt{2}x)\\ -3\sin(\sqrt{2}x)\end{bmatrix}e^{3x}\end{aligned}$$

. This is the fully general, *real* solution to the original set of DEs.

Non-homogeneous case

of

$$\mathbf{y}' - \mathbb{A}\mathbf{y} = \mathbf{r}(x)$$

where $\mathbf{r}$ is a given *vector* function of x (effectively n functions, one for each equation). We already know how to solve the corresponding homogeneous version.

There are two techniques to find a particular solution $\mathbf{y}^{(p)}$ to the complete equation; the general solution is then constructed in the usual

$$c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + \ldots + c_n\mathbf{y}^{(n)} + \mathbf{y}^{(p)}$$

manner.

The first of these techniques (for constructing y_p) is:

Variation of Parameters

As a *trial solution*, we use

$$\mathbf{y}^{(T)} = \mathbb{Y}\mathbf{c}$$

where $\mathbb{Y}$ is an n by n matrix of functions, with the n basic solutions of the homogeneous equation comprising its individual *columns*, thus:

$$\mathbb{Y} \equiv \begin{array}{|c|c|c|c|}\hline \mathbf{y}^{(1)} & \mathbf{y}^{(2)} & \ldots & \mathbf{y}^{(n)} \\ \hline \end{array}$$

and $\mathbf{c}$ being a single column of the c_i coefficients: $\mathbf{c} \equiv \begin{array}{|c|}\hline c_1 \\ \hline c_2 \\ \hline \vdots \\ \hline c_n \\ \hline \end{array}$, each now considered a *function* of x [$\mathbb{Y}\mathbf{c}$ is just a matrix representation of $c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} + \ldots + c_n\mathbf{y}^{(n)}$, with the c_i coefficients now being 'variable'].

Substituting in the full (non-homogeneous) equation, and realizing that $\mathbb{Y}' \equiv \mathbb{A}\mathbb{Y}$, we obtain: $\mathbb{Y}' \cdot \mathbf{c} + \mathbb{Y} \cdot \mathbf{c}' - \mathbb{A} \cdot \mathbb{Y} \cdot \mathbf{c} = r \Rightarrow \mathbf{c}' = \mathbb{Y}^{-1} \cdot \mathbf{r}$. Integrating the right hand side (component by component) yields $\mathbf{c}$. The particular solution is thus

$$\mathbf{y}^{(p)} = \mathbb{Y} \int \mathbb{Y}^{-1} \cdot \mathbf{r}(x)\, dx$$

EXAMPLE:

$$\mathbf{y}' = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & 2 \\ \hline \end{array}\,\mathbf{y} + \begin{array}{|c|}\hline 4e^{5x} \\ \hline 0 \\ \hline \end{array}$$

$\Rightarrow \lambda^2 - 5\lambda + 4 = 0 \Rightarrow \lambda_{1,2} = 1$ and 4 with the respective eigenvectors [easy to construct]: $\begin{array}{|c|}\hline 1 \\ \hline -1 \\ \hline\end{array}$ and $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$.

Thus $\mathbb{Y} = \begin{array}{|c|c|}\hline e^x & 2e^{4x} \\ \hline -e^x & e^{4x} \\ \hline\end{array} \Rightarrow \mathbb{Y}^{-1} = \begin{array}{|c|c|}\hline \frac{1}{3}e^{-x} & -\frac{2}{3}e^{-x} \\ \hline \frac{1}{3}e^{-4x} & \frac{1}{3}e^{-4x} \\ \hline\end{array}$. This matrix, multiplied by $\mathbf{r}(x)$, yields $\begin{array}{|c|}\hline \frac{4}{3}e^{4x} \\ \hline \frac{4}{3}e^{x} \\ \hline\end{array}$. The componentwise integration of the last vector is trivial: $\begin{array}{|c|}\hline \frac{1}{3}e^{4x} \\ \hline \frac{4}{3}e^{x} \\ \hline\end{array}$, (pre)multiplied by $\mathbb{Y}$ finally results in: $\mathbf{y}^{(p)} = \begin{array}{|c|}\hline 3e^{5x} \\ \hline e^{5x} \\ \hline\end{array}$. The general solution is thus $\mathbf{y} = c_1 \begin{array}{|c|}\hline 1 \\ \hline -1 \\ \hline\end{array}\ e^x + c_2 \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}\ e^{4x} + \begin{array}{|c|}\hline 3e^{5x} \\ \hline e^{5x} \\ \hline\end{array}$.

Let us make this into an initial-value problem: $y_1(0) = 1$ and $y_2(0) = -1 \Leftrightarrow \mathbf{y}(0) = \mathbb{Y}(0)\mathbf{c} + \mathbf{y}^{(p)}(0) = \begin{array}{|c|}\hline 1 \\ \hline -1 \\ \hline\end{array}$.

Solving for $\mathbf{c} = \begin{array}{|c|c|}\hline \frac{1}{3} & -\frac{2}{3} \\ \hline \frac{1}{3} & \frac{1}{3} \\ \hline\end{array} \left(\begin{array}{|c|}\hline 1 \\ \hline -1 \\ \hline\end{array} - \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline\end{array} \right) = \begin{array}{|c|}\hline \frac{2}{3} \\ \hline -\frac{4}{3} \\ \hline\end{array} \Rightarrow \mathbf{y} = \begin{array}{|c|}\hline \frac{2}{3} \\ \hline -\frac{2}{3} \\ \hline\end{array}\ e^x - \begin{array}{|c|}\hline \frac{8}{3} \\ \hline \frac{4}{3} \\ \hline\end{array}\ e^{4x} + \begin{array}{|c|}\hline 3e^{5x} \\ \hline e^{5x} \\ \hline\end{array}$.

Undetermined coefficients:
Works only for two *special cases* of $\mathbf{r}(x)$
$\triangleright$ When the non-homogeneous part of the equation has the form of

$$\left(\mathbf{a}_k x^k + \mathbf{a}_{k-1} x^{k-1} + \ldots + \mathbf{a}_1 x + \mathbf{a}_0\right) e^{\beta\, x}$$

we use the following 'trial' solution (which is guaranteed to work) to construct $\mathbf{y}^{(p)}$:

$$\left(\mathbf{b}_m x^m + \mathbf{b}_{m-1} x^{m-1} + \ldots + \mathbf{b}_1 x + \mathbf{b}_0\right) e^{\beta\, x}$$

where m equals k *plus* the number of levels of β as an eigenvalue of $\mathbb{A}$ (if β is not an eigenvalue, $m = k$.

When β does *not* coincide with any *eigenvalue* of $\mathbb{A}$, the equations to solve to obtain $\mathbf{b}_k$, $\mathbf{b}_{k-1}$, ..., $\mathbf{b}_1$ are

$$\begin{aligned}
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_k &= -\mathbf{a}_k \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k-1} &= k\mathbf{b}_k-\mathbf{a}_{k-1} \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k-2} &= (k-1)\mathbf{b}_{k-1}-\mathbf{a}_{k-2} \\
&\vdots \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_0 &= \mathbf{b}_1-\mathbf{a}_0
\end{aligned}$$

Since $\mathbb{A}-\beta\mathbb{I}$ is a *regular* matrix (having an inverse), solving these is quite routine (as long as we start from the top).

When β coincides with a *simple* (as opposed to multiple) eigenvalue of $\mathbb{A}$, we have to solve

$$\begin{aligned}
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k+1} &= \mathbf{0} \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_k &= (k+1)\mathbf{b}_{k+1}-\mathbf{a}_k \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_{k-1} &= k\mathbf{b}_k-\mathbf{a}_{k-1} \\
&\vdots \\
(\mathbb{A}-\beta\mathbb{I})\,\mathbf{b}_0 &= \mathbf{b}_1-\mathbf{a}_0
\end{aligned}$$

Thus, $\mathbf{b}_{k+1}$ must be the corresponding eigenvector, multiplied by such a constant as to make the second equation solvable [remember that now $(\mathbb{A}-\beta\mathbb{I})$ is *singular*]. Similarly, when solving the second equation for $\mathbf{b}_k$, a c-multiple of the same eigenvector must be added to the solution, with c chosen so that the third equation is solvable, etc. Each $\mathbf{b}_i$ is thus *unique*, even though finding it is rather tricky.

We will not try extending this procedure to the case of β being a *double* (or multiple) eigenvalue of $\mathbb{A}$.

$\triangleright$ On the other hand, the extension to the case of

$$\mathbf{r}(x)=\mathbf{P}(x)e^{px}\cos(qx)+\mathbf{Q}(x)e^{px}\sin(qx)$$

where $\mathbf{P}(x)$ and $\mathbf{Q}(x)$ are polynomials in x (with vector coefficients), and $p+i\,q$ is *not* an eigenvalue of $\mathbb{A}$ is quite simple: The trial solution has the same form as $\mathbf{r}(x)$, except that the two polynomials will have undetermined coefficients, and will be of the *same* degree (equal to the degree of $\mathbf{P}(x)$ or $\mathbf{Q}(x)$, whichever is *larger*). This trial solution is then substituted into the full equation, and the coefficients of each power of x are matched, separately for the $\cos(qx)$-proportional and $\sin(qx)$-proportional terms.

In addition, one can also use the superposition principle [i.e. dividing $\mathbf{r}(x)$ into two or more parts, getting a particular solution for each part separately, and then adding them all up].

EXAMPLES:

1.

$$\mathbf{y}' = \begin{bmatrix} -4 & -4 \\ 1 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2x} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} e^{-x}$$

$\Rightarrow \lambda^2 + 2\lambda - 4 = 0 \Rightarrow \lambda_{1,2} = -1 \pm \sqrt{5}$. We already know how to construct the solution to the homogeneous part of the equation, we show only how to deal with $\mathbf{y}^{(p)} = \mathbf{y}^{(p_1)} + \mathbf{y}^{(p_2)}$ [for each of the two $\mathbf{r}(x)$ terms]:

$\mathbf{y}^{(p_1)} = \mathbf{b}e^{2x}$, substituted back into the equation gives

$$\begin{bmatrix} -6 & -4 \\ 1 & 0 \end{bmatrix} \mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$\Rightarrow \mathbf{b} = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}$.

Similarly $\mathbf{y}^{(p_2)} = \mathbf{b}e^{-x}$ [a different $\mathbf{b}$], substituted, gives

$$\begin{bmatrix} -3 & -4 \\ 1 & 3 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$\Rightarrow \mathbf{b} = \begin{bmatrix} -\frac{8}{5} \\ \frac{6}{5} \end{bmatrix}$.

The full particular solution is thus

$$\mathbf{y}^{(p)} = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} e^{2x} + \begin{bmatrix} -\frac{8}{5} \\ \frac{6}{5} \end{bmatrix} e^{-x}$$

2.

$$\mathbf{y}' = \begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{x} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

$\Rightarrow \lambda^3 - \lambda^2 - 24\lambda - 7 = 0$. If we are interested in the particular solution *only*, we need to check that neither $\beta = 1$ nor $\beta = 0$ are the roots of the characteristic polynomial [true].

Thus $\mathbf{y}^{(p_1)} = \mathbf{b}e^x$ where $\mathbf{b}$ solves

$$\begin{bmatrix} -2 & 2 & 3 \\ 5 & -2 & -2 \\ 5 & 3 & 2 \end{bmatrix} \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \mathbf{b} = \begin{bmatrix} -\frac{2}{31} \\ \frac{20}{31} \\ -\frac{25}{31} \end{bmatrix}$.

Similarly $\mathbf{y}^{(p_2)} = \mathbf{b}$ where

$$\begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}$$

$\Rightarrow \mathbf{b} = \begin{bmatrix} -\frac{12}{7} \\ \frac{72}{7} \\ \frac{52}{7} \end{bmatrix}$.

Answer:

$$\mathbf{y}^{(p)} = \begin{bmatrix} -\frac{2}{31} \\ \frac{20}{31} \\ -\frac{25}{31} \end{bmatrix} e^x + \begin{bmatrix} -\frac{12}{7} \\ \frac{72}{7} \\ \frac{52}{7} \end{bmatrix}$$

3.

$$\mathbf{y}' = \begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} x-1 \\ 2 \\ -2x \end{bmatrix}$$

[characteristic polynomial same as previous example]. $\mathbf{y}^{(p)} = \mathbf{b}_1 x + \mathbf{b}_0$ with

$$\begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{b}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

implying

$$\mathbf{b}_1 = \begin{bmatrix} -\frac{5}{7} \\ \frac{51}{7} \\ -\frac{38}{7} \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 2 & 3 \\ 5 & -1 & -2 \\ 5 & 3 & 3 \end{bmatrix} \mathbf{b}_0 = \begin{bmatrix} -\frac{5}{7} \\ \frac{51}{7} \\ -\frac{38}{7} \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

implying

$$\mathbf{b}_0 = \begin{bmatrix} \frac{155}{7} \\ -\frac{1210}{7} \\ \frac{863}{7} \end{bmatrix}$$

Thus

$$\mathbf{y}^{(p)} = \begin{bmatrix} \frac{5}{7}x + \frac{155}{7} \\ \frac{51}{7}x - \frac{1210}{7} \\ -\frac{38}{7}x + \frac{863}{7} \end{bmatrix}$$

4.

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-x}$$

$\Rightarrow \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = -1,\ -2$. Now our $\beta = -1$ 'coincides' with a simple eigenvalue.

$\mathbf{y}^{(p)} = (\mathbf{b}_1 x + \mathbf{b}_0)\, e^{-x}$ where

$$\begin{bmatrix} -3 & -3 \\ 2 & 2 \end{bmatrix} \mathbf{b}_1 = \mathbf{0}$$

implying

$$\mathbf{b}_1 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} -3 & -3 \\ 2 & 2 \end{bmatrix} \mathbf{b}_0 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc||cc} -3 & -3 & 1 & -1 \\ 2 & 2 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc||cc} 1 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 8 \end{array}\right]$$

This fixes the value of c_1 at $-8 \Rightarrow$

$$\mathbf{b}_1 = \begin{array}{|c|}\hline -8 \\ \hline 8 \\ \hline\end{array}$$

and

$$\mathbf{b}_0 = \begin{array}{|c|}\hline 3 \\ \hline 0 \\ \hline\end{array} + c_0 \begin{array}{|c|}\hline 1 \\ \hline -1 \\ \hline\end{array}$$

Being the last $\mathbf{b}$, we can set $c_0 = 0$ (not to duplicate the homogeneous part of the solution).

Answer:

$$\mathbf{y}^{(p)} = \begin{array}{|c|}\hline -8x+3 \\ \hline 8x \\ \hline\end{array} e^{-x}$$

5.

$$\mathbf{y}' = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 4 & -1 \\ \hline -5 & -8 & -3 \\ \hline\end{array} \mathbf{y} + \begin{array}{|c|}\hline x \\ \hline 0 \\ \hline 4 \\ \hline\end{array}$$

$\Rightarrow \lambda^3 - 2\lambda^2 - 15\lambda = 0 \Rightarrow \beta = 0$ is a simple eigenvalue.

We construct $\mathbf{y}^{(p)} = \mathbf{b}_2 x^2 + \mathbf{b}_1 x + \mathbf{b}_0$ where

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 4 & -1 \\ \hline -5 & -8 & -3 \\ \hline\end{array} \mathbf{b}_2 = \mathbf{0}$$

implying

$$\mathbf{b}_2 = c_2 \begin{array}{|c|}\hline -5 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 4 & -1 \\ \hline -5 & -8 & -3 \\ \hline\end{array} \mathbf{b}_1 = 2\mathbf{b}_2 - \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline 0 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c||c|c|}\hline 1 & 1 & 1 & -10 & -1 \\ \hline 1 & 4 & -1 & 4 & 0 \\ \hline -5 & -8 & -3 & 6 & 0 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c||c|c|}\hline 1 & 0 & \frac{5}{3} & -\frac{44}{3} & -\frac{4}{3} \\ \hline 0 & 1 & -\frac{2}{3} & \frac{14}{3} & \frac{1}{3} \\ \hline 0 & 0 & 0 & 1 & \frac{2}{15} \\ \hline\end{array}$$

implying $c_2 = -\frac{2}{15}$, and

$$\left[\begin{array}{ccc||c} 1 & 0 & \frac{5}{3} & \frac{28}{45} \\ 0 & 1 & -\frac{2}{3} & -\frac{13}{45} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

implying

$$\mathbf{b}_1 = \begin{bmatrix} \frac{28}{45} \\ -\frac{13}{45} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & -1 \\ -5 & -8 & -3 \end{bmatrix} \mathbf{b}_0 = \mathbf{b}_1 - \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc||cc} 1 & 1 & 1 & -5 & \frac{28}{45} \\ 1 & 4 & -1 & 2 & -\frac{13}{45} \\ -5 & -8 & -3 & 3 & -\frac{180}{45} \end{array}\right]$$

$$\left[\begin{array}{ccc||cc} 1 & 0 & \frac{5}{3} & -\frac{22}{3} & \frac{125}{135} \\ 0 & 1 & -\frac{2}{3} & \frac{7}{3} & -\frac{41}{135} \\ 0 & 0 & 0 & -15 & -\frac{81}{45} \end{array}\right]$$

implying $c_1 = -\frac{3}{25}$ and

$$\left[\begin{array}{ccc||c} 1 & 0 & \frac{5}{3} & \frac{1219}{675} \\ 0 & 1 & -\frac{2}{3} & -\frac{394}{675} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$

$$\mathbf{b}_0 = \begin{bmatrix} \frac{1219}{675} \\ -\frac{394}{675} \\ 0 \end{bmatrix}$$

[no need for c_0].

Answer:

$$\mathbf{y}^{(p)} = \begin{bmatrix} \frac{2}{3}x^2 + \frac{11}{9}x + \frac{1219}{675} \\ -\frac{4}{15}x^2 - \frac{119}{225}x - \frac{394}{675} \\ -\frac{2}{5}x^2 - \frac{9}{25}x \end{bmatrix}$$

Final Remark:

Note that an equation of the type

$$\mathbb{B}\mathbf{y}' = \mathbb{A}\mathbf{y} + \mathbf{r}$$

can be converted to the regular type by (pre)multiplying it by $\mathbb{B}^{-1}$.

EXAMPLE:

Power-Series Solution of

$$y'' + f(x)y' + g(x)y = 0$$

i.e. seeking a solution in the form of

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$$

The main idea

is to substitute this expression into the differential equation, and make the coefficient of *each* power of x cancel.

This results in (infinitely many, but regular) equations for the unknown coefficients c_0, c_1, c_2, ...

These can usually be solved in a recurrent [some call it recursive] manner (i.e. by deriving a simple formula which computes c_k based on c_0, c_1, ..., c_{k-1},where c_0 and c_1 can normally be chosen *arbitrarily*).

EXAMPLES:

1.

$$y'' + y = 0$$

implying

$$\sum_{i=2}^{\infty} i(i-1)c_i x^{i-2} + \sum_{i=0}^{\infty} c_i x^i \equiv 0$$

The main thing is to express the left hand side as a *single* infinite summation, by replacing the index i of the first term by $i^* + 2$, thus: $\sum\limits_{i^*=0}^{\infty} (i^* + 2)(i^* + 1)c_{i^*+2} x^{i^*}$ [note that the lower limit had to be adjusted accordingly]. But i^*

is just a dummy index which can be called j, k or anything else *including* i. This way we get (combining both terms):

$$\sum_{i=0}^{\infty} \left[(i+2)(i+1)c_{i+2} + c_i\right] x^i \equiv 0$$

which implies that the expression in square brackets must be identically equal to zero. This yields the following recurrent formula

$$c_{i+2} = \frac{-c_i}{(i+2)(i+1)}$$

where $i = 0, 1, 2, ...$, from which we can easily construct the compete sequence of the c-coefficients, as follows: Starting with c_0 arbitrary, we get

$$\begin{aligned}
c_2 &= \frac{-c_0}{2 \times 1} \\
c_4 &= \frac{-c_2}{4 \times 3} = \frac{c_0}{4!} \\
c_6 &= \frac{-c_4}{6 \times 5} = \frac{-c_0}{6!} \\
&\vdots \\
c_{2k} &= \frac{(-1)^k}{(2k)!} c_0
\end{aligned}$$

Similarly, choosing an arbitrary value for c_1 we get

$$\begin{aligned}
c_3 &= \frac{-c_1}{3!} \\
c_5 &= \frac{c_1}{5!} \\
&\vdots
\end{aligned}$$

The complete solution is thus

$$y = c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right) + c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)$$

where the infinite expansions can be easily identified as those of $\cos x$ and $\sin x$, respectively. We have thus obtained the expected $y = c_0 \cos x + c_1 \sin x$ [check].

We will not always be lucky enough to identify each solution as a combination of simple functions, but we should be able to recognize at least the following:

$$\begin{aligned}
(1-ax)^{-1} &= 1+ax+a^2x^2+a^3x^3+... \\
e^{ax} &= 1+ax+\frac{a^2x^2}{2!}+\frac{a^3x^3}{3!}+... \\
\ln(1-ax) &= -ax-\frac{a^2x^2}{2}-\frac{a^3x^3}{3}-...
\end{aligned}$$

with a being any number [often $a=1$].

Also, realize that

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+...$$

[a power of x missing] must be $\frac{\sin x}{x}$,

$$1-\frac{3x^2}{2!}+\frac{9x^4}{4!}-\frac{27x^6}{6!}+...$$

is the expansion of $\cos(\sqrt{3}x)$,

$$1+\,x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+...$$

must be $\exp(x^2)$, and

$$1-\frac{x}{3!}+\frac{x^2}{5!}-\frac{x^3}{7!}+...$$

is $\frac{\sin\sqrt{x}}{\sqrt{x}}$.

2.

$$(1-x^2)y''-2xy'+2y=0$$

implying

$$\begin{aligned}
&\sum_{i=2}^{\infty} i(i-1)c_i x^{i-2}-\sum_{i=2}^{\infty} i(i-1)c_i x^i \\
&-2\sum_{i=1}^{\infty} ic_i x^i+2\sum_{i=0}^{\infty} c_i x^i \equiv 0
\end{aligned}$$

By reindexing (to get the same x^i in each term) we get

$$\sum_{i^*=0}^{\infty}(i^*+2)(i^*+1)c_{i^*+2}x^{i^*} - \sum_{i=2}^{\infty} i(i-1)c_i x^i$$
$$-2\sum_{i=1}^{\infty} ic_i x^i + 2\sum_{i=0}^{\infty} c_i x^i \equiv 0$$

Realizing that, as a dummy index, i^* can be called i (this is the last time we introduced i^*, form now on we will call it i directly), our equation becomes:

$$\sum_{i=0}^{\infty}\left[(i+2)(i+1)c_{i+2} - i(i-1)c_i - 2ic_i + 2c_i\right] x^i \equiv 0$$

(we have adjusted the lower limit of the second and third term down to 0 *without* affecting the answer – careful with this though, things are not always that simple).

The square brackets must be identically equal to zero which implies:

$$c_{i+2} = \frac{i^2+i-2}{(i+2)(i+1)}\, c_i = \frac{i-1}{i+1}\, c_i$$

where $i = 0,\ 1,\ 2,\ ....$ Starting with an arbitrary c_0 we get

$$\begin{aligned}
c_2 &= -c_0 \\
c_4 &= \frac{1}{3}c_2 = -\frac{1}{3}c_0 \\
c_6 &= \frac{3}{4}c_4 = -\frac{1}{4}c_0 \\
c_6 &= -\frac{1}{6}c_0 \\
&....
\end{aligned}$$

Starting with c_1 we get

$$\begin{aligned}
c_3 &= 0 \\
c_5 &= 0c_7 = 0 \\
&....
\end{aligned}$$

The solution is thus

$$y = c_0(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - ...) + c_1 x$$

One of the basic solutions is thus simply equal to x, once we know that we can use the V of P technique to get an analytic expression for the other solution (try it), but with a bit of juggling we can easily figure out that the second basic solution is

$$1 - \frac{x}{2}\ln\frac{1+x}{1-x}$$

3.

$$y'' - 3y' + 2y = 0$$

implying

$$\sum_{i=2}^{\infty} i(i-1)c_i x^{i-2} - 3\sum_{i=1}^{\infty} i c_i x^{i-1} + 2\sum_{i=0}^{\infty} c_i x^i \equiv 0$$

$$\sum_{i=0}^{\infty} \left[(i+2)(i+1)c_{i+2} - 3(i+1)c_{i+1} + 2c_i\right] x^i \equiv 0$$

$\Rightarrow$

$$c_{i+2} = \frac{3(i+1)c_{i+1} - 2c_i}{(i+2)(i+1)}$$

By choosing $c_0 = 1$ and $c_1 = 0$ we can generate the *first* basic solution:

$$c_0(1 - x^2 - x^3 - \frac{7}{12}x^4 - \frac{1}{4}x^5 - ...)$$

similarly with $c_0 = 0$ and $c_1 = 1$ the *second* basic solution is:

$$c_1(x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{5}{8}x^4 + \frac{31}{120}x^5 + ...)$$

There is no obvious pattern to either sequence of coefficients. Yet we know that, in this case, the two basic solutions should be simply e^x and e^{2x}. The trouble is that our power-series technique presents these in a hopelessly entangled form of $2e^x - e^{2x}$ [*our* first basic solution] and $e^{2x} - e^x$ [the second], and we have no way of properly separating them.

Sometimes the *initial conditions* may help, e.g. $y(0) = 1$ and $y'(0) = 1$ [these are effectively the values of c_0 and c_1, respectively], leading to

$$\begin{aligned} c_2 &= \frac{3-2}{2} = \frac{1}{2} \\ c_3 &= \frac{3-2}{3 \times 2} = \frac{1}{6} \\ c_4 &= \frac{\frac{3}{2}-1}{4 \times 3} = \frac{1}{24} \\ &\ldots.. \end{aligned}$$

form which the pattern of the e^x-expansion clearly emerges. We can then *conjecture* that $c_i = \frac{1}{i!}$ and *prove it* by substituting into the previous recurrence formula.

Similarly, the initial values of $y(0) = c_0 = 1$ and $y'(0) = c_1 = 2$ will lead to

$$1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{3!} + \ldots$$

[the expansion of e^{2x}]. Prove that $c_i = \frac{2^i}{i!}$ is also a solution of our recurrence equation!

More examples (introducing new, **special functions**):

Legendre Equation

$$(1 - x^2)y'' - 2xy' + \lambda y = 0$$

(Note that we already solved this equation with $\lambda = 2$, see Example 2 of the 'main idea' section).

The expression to be identically equal to zero is

$$(i+2)(i+1)c_{i+2} - i(i-1)c_i - 2ic_i + \lambda c_i$$

$\Rightarrow$

$$c_{i+2} = -\frac{\lambda - (i+1)\, i}{(i+2)(i+1)} c_i.$$

Suppose we allow only *polynomial* solutions. This will happen only if the numerator of the $c_{i+2} = \ldots$ formula is zero for some integer value of i, i.e. iff

$$\lambda = (n+1)\, n$$

(for n even, we would have to choose $c_0 = 1$ and $c_1 = 0$, for n odd reverse).

We then get the following polynomial solutions – $P_n(x)$ being the standard notation:

$$\begin{aligned}
P_0(x) &\equiv 1 \\
P_1(x) &= x \\
P_2(x) &= 1 - 3x^2 \\
P_3(x) &= x - \frac{5}{3}x^3 \\
P_4(x) &= 1 - 10x^2 + \frac{35}{3}x^4 \\
& \ldots.
\end{aligned}$$

These are the so called Legendre polynomials (important in Physics and other fields).

The corresponding second basic solution (one can find it via V of P - it is no longer a polynomial) is called Legendre function of second kind.

Chebyshev equation

$$(1 - x^2)y'' - xy' + \lambda y = 0$$

It easy to see that the Legendre recurrence formula needs to be modified to read

$$(i + 2)(i + 1)c_{i+2} - i(i - 1)c_i - ic_i + \lambda c_i = 0$$

$\Rightarrow$

$$c_{i+2} = -\frac{\lambda - i^2}{(i + 2)(i + 1)}c_i$$

To make the expansion finite (polynomial) we have to choose

$$\lambda = n^2$$

This leads to the following Chebyshev polynomials:

$$\begin{aligned}
T_0 &\equiv 1 \\
T_1 &= x \\
T_2 &= 1 - 2x^2 \\
T_3 &= x - \frac{4}{3}x^3 \\
& \ldots.
\end{aligned}$$

Method of Frobenius

The power-series technique described so far is applicable only when both $f(x)$ and $g(x)$ of the main equation can be expanded at $x = 0$. This condition is violated when either f or g (or both) involve a division by x or its power [e.g. $y'' + \frac{1}{x}\, y' + (1 - \frac{1}{4x^2})\, y = 0$].

To make the power-series technique work in some of these cases, we must extend it in a manner described shortly (the extension is called the method of Frobenius).

The only restriction is that the singularity of f is (at most) of the first degree in x, and that of g is no worse than of the second degree. We can thus rewrite the main equation as

$$y'' + \frac{a(x)}{x} y' + \frac{b(x)}{x^2} y = 0$$

where $a(x)$ and $b(x)$ are *regular* [i.e. 'expandable': $a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$, and $b(x) = b_0 a_0 + b_1 x + b_2 x^2 + b_3 x^3 + ...$].

The trial solution now has the form of

$$y^{(T)} = \sum_{i=0}^{\infty} c_i x^{r+i} = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + ...$$

where r is a number (no necessarily an integer) yet to be found.

When substituted into the above differential equation (which is normally simplified by multiplying it by x^2), the overall coefficient of the lowest (r^{th}) power of x is

$$[r(r-1) + a_0 r + b_0]\, c_0$$

This must (as all the other coefficients) be equal to zero, yielding the so called indicial equation for r

$$r^2 + (a_0 - 1)r + b_0 = 0$$

Even after ignoring the possibility of complex roots (assume this never happens to us), we have to categorize the solution of the indicial (simple quadratic) equation into *three* separate cases:

1. *Two distinct* real roots which don't differ by an integer

2. A *double* root

3. Two roots which *differ* by an *integer*, i.e. $r_2 - r_1$ is a nonzero integer (zero is covered by Case 2).

We have to develop our technique separately for each of the three cases:

Distinct Real Roots

The trial solution is substituted into the differential equation with r having the value of one of the roots of the indicial equation. Making the coefficients of each power of x cancel out, one gets the usual recurrence formula for the sequence of the c-coefficients [actually, *two* such *sequences*, one with the first root r_1 and the other, say c_i^*, with r_2]. Each of the two recurrence formula allows a free choice of the first c (called c_0 and c_0^*, respectively); the rest of each sequence must uniquely follow.

EXAMPLE:

$$x^2y'' + (x^2 + \frac{5}{36})y = 0$$

[later on we will see that this is a special case of the so called Bessel equation].

Since $a(x) \equiv 0$ and $b(x) = x^2 + \frac{5}{36}$ the indicial equation reads

$$r^2 - r + \frac{5}{36} = 0$$

$\Rightarrow r_{1,2} = \frac{1}{6}$ and $\frac{5}{6}$ [Case 1].

Substituting our trial solution into the differential equation yields

$$\sum_{i=0}^{\infty} c_i(r+i)(r+i-1)x^{r+i} + \frac{5}{36}\sum_{i=0}^{\infty} c_i x^{r+i} + \sum_{i=0}^{\infty} c_i x^{r+i+2} = 0$$

. Introducing a new dummy index $i^* = i + 2$ we get

$$\sum_{i=0}^{\infty} c_i[(r+i)(r+i-1) + \frac{5}{36}]x^{r+i} + \sum_{i^*=2}^{\infty} c_{i^*-2}x^{r+i^*} = 0$$

Before we can combine the two sums together, we have to deal with the exceptional $i = 0$ and 1 terms. The first ($i = 0$) term gave us our indicial equation and was made to disappear by taking r to be one of the equation's two roots. The second one has the coefficient of $c_1[(r+1)r+\frac{5}{36}]$ which can be eliminated only by $c_1 \equiv 0$. The rest of the left hand side is

$$\sum_{i=0}^{\infty}\left\{c_i[(r+i)(r+i-1)+\frac{5}{36}]+c_{i-2}\right\}x^{r+i}$$

$\Rightarrow$

$$c_i = \frac{-c_{i-2}}{(r+i)(r+i-1)+\frac{5}{36}}$$

So far we have avoided substituting a specific root for r [to be able to deal with both cases at the same time], now, to build our two basic solutions, we have to set

1. $r=\frac{1}{6}$, getting

$$c_i = \frac{-c_{i-2}}{i\,(i-\frac{2}{3})}$$

$\Rightarrow$

$$\begin{aligned} c_2 &= \frac{-c_0}{2\times\frac{4}{3}} \\ c_4 &= \frac{c_0}{4\times 2\times\frac{4}{3}\times\frac{10}{3}} \\ c_6 &= \frac{-c_0}{6\times 4\times 2\times\frac{4}{3}\times\frac{10}{3}\times\frac{16}{3}} \\ &\ldots\ldots \end{aligned}$$

[the odd-indexed coefficients must be all equal to zero].

Even though the expansion has an obvious pattern, the function cannot be identified as a 'known' function. Based on this expansion, one can *introduce* a new function [eventually a whole bunch of them], called Bessel functions, as we do in full detail later on.

For those of you who know the Γ-function, the solution can be expressed in a more compact form of

$$\tilde{c}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{x}{2})^{2k+1/6}}{k!\Gamma(k+\frac{2}{3})}$$

2. $r = \frac{5}{6}$, getting

$$c_i^* = \frac{-c_{i-2}^*}{i\,(i+\frac{2}{3})}$$

$\Rightarrow$

$$\begin{aligned} c_2^* &= \frac{-c_0^*}{2 \times \frac{8}{3}} \\ c_4^* &= \frac{c_0^*}{4 \times 2 \times \frac{8}{3} \times \frac{14}{3}} \\ c_6^* &= \frac{-c_0^*}{6 \times 4 \times 2 \times \frac{8}{3} \times \frac{14}{3} \times \frac{20}{3}} \\ &\quad \ldots.. \end{aligned}$$

Double root

The first basic solution is constructed in the usual manner of

$$y_1 = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + ...$$

The second basic solution has the form [guaranteed to work] of:

$$y_2 = y_1 \ln x + c_0^* x^r + c_1^* x^{r+1} + c_2^* x^{r+2} + ...$$

where y_1 is the first basic solution (with c_0 set equal to 1, i.e. removing the multiplicative constant). The corresponding recurrence formula (for the c_i^*'s) will offer us a 'free choice' of c_0^*, which we normally set equal to 0 (a nonzero choice would only add $c_0^* y_1$ to our second basic solution). After that, the rest of the c_i^*'s uniquely follows (they may turn out to be all 0 in some cases).

EXAMPLES:

(1)

$$(1+x)x^2 y'' - (1+2x)xy' + (1+2x)y = 0$$

[$a(x) = -\frac{1+2x}{1+x}$ and $b(x) = \frac{1+2x}{1+x}$]. The indicial equation is

$$r^2 - 2r + 1 = 0$$

$\Rightarrow r_{1,2} = 1 \pm 0$ [double]. Substituting $\sum\limits_{i=0}^{\infty} c_i x^{i+1}$ for y yields

$$\sum_{i=0}^{\infty} c_i(i+1)ix^{i+1} + \sum_{i=0}^{\infty} c_i(i+1)ix^{i+2} - \sum_{i=0}^{\infty} c_i(i+1)x^{i+1}$$
$$-2\sum_{i=0}^{\infty} c_i(i+1)x^{i+2} + \sum_{i=0}^{\infty} c_i x^{i+1} + 2\sum_{i=0}^{\infty} c_i x^{i+2} = 0$$

Combining terms with like powers of x:

$$\sum_{i=0}^{\infty} c_i i^2 x^{i+1} + \sum_{i=0}^{\infty} c_i i(i-1)x^{i+2} = 0$$

Adjusting the index of the second sum:

$$\sum_{i=0}^{\infty} c_i i^2 x^{i+1} + \sum_{i=1}^{\infty} c_{i-1}(i-1)(i-2)x^{i+1} = 0$$

The 'exceptional' $i = 0$ term must equal to zero automatically, our indicial equation takes care of that [check], the rest implies

$$c_i = -\frac{(i-1)(i-2)}{i^2} c_{i-1}$$

for $i = 1, 2, 3, ...$, yielding $c_1 = 0$, $c_2 = 0$,.... The first basic solution is thus $c_0 x$ [i.e. $y_1 = x$, verify!]. Once we have identified the first basic solution as a simple function [when lucky] we have *two* options:

(a) Use V of P:

$$y(x) = c(x) \cdot x$$

$\Rightarrow$

$$(1+x)xc'' + c' = 0$$

$\Rightarrow$

$$\frac{dz}{z} = \left(\frac{1}{1+x} - \frac{1}{x}\right) dx$$

$\Rightarrow$

$$\ln z = \ln(1+x) - \ln x + \tilde{c}$$

$\Rightarrow$

$$c' = c_0^* \frac{1+x}{x}$$

$\Rightarrow$

$$c(x) = c_0^*(\ln x + x) + c_0$$

This makes it clear that the second basic solution is

$$x \ln x + x^2$$

(b) Insist on using Frobenius: Substitute

$$y^{(T)} = x \ln x + \sum_{i=0}^{\infty} c_i^* x^{i+1}$$

into the original equation. The sum will give you the same contribution as before, the $x \ln x$ term (having no unknowns) yields an extra, non-homogeneous term of the corresponding recurrence equation. There is a bit of an automatic simplification when substituting $y_1 \ln x$ (our $x \ln x$) into the equation, as the $\ln x$-proportional terms must cancel. What we need is thus $y \to 0$, $y' \to \frac{y_1}{x}$ and $y'' \to 2\frac{y_1'}{x} - \frac{y_1}{x^2}$ .This substitution results in the same old [except for $c \to c^*$]

$$\sum_{i=0}^{\infty} c_i^* i^2 x^{i+1} + \sum_{i=1}^{\infty} c_{i-1}^* (i-1)(i-2) x^{i+1}$$

on the left hand side of the equation, and

$$-(1+x)x^2 \cdot \frac{1}{x} + (1+2x)x = x^2$$

[don't forget to reverse the sign] on the right hand side. This yields the same set of recurrence formulas as before, *except* at $i = 1$ [due to the nonzero right-hand-side term]. Again we get a 'free choice' of c_0^* [indicial equation takes care of that], which we utilize by setting c_0^* equal to *zero* (or anything which simplifies the answer), since a nonzero c_0^* would only add a redundant $c_0^* y_1$ to our *second* basic solution. The x^2-part of the equation ($i = 1$) then reads:

$$c_1^* x^2 + 0 = x^2$$

$\Rightarrow c_1^* = 1$. The rest of the sequence follows from

$$c_i^* = -\frac{(i-1)(i-2)}{i^2} c_{i-1}^*,$$

$i = 2, 3, 4... \Rightarrow c_2^* = c_3^* = ... = 0$ as before. The second basic solution is thus

$$y_1 \ln x + c_1^* x^2 = x \ln x + x^2$$

[check].

(2)

$$x(x-1)y'' + (3x-1)y' + y = 0$$

$[a(x) = \frac{3x-1}{x-1}\ b(x) = \frac{x}{x-1}] \Rightarrow r^2 = 0$ [double root of 0]. Substituting $y^{(T)} = \sum_{i=0}^{\infty} c_i x^{i+0}$ yields

$$\sum_{i=0}^{\infty} i(i-1)c_i x^i - \sum_{i=0}^{\infty} i(i-1)c_i x^{i-1} + 3\sum_{i=0}^{\infty} ic_i x^i$$
$$- \sum_{i=0}^{\infty} ic_i x^{i-1} + \sum_{i=0}^{\infty} c_i x^i = 0$$

$\Leftrightarrow$

$$\sum_{i=0}^{\infty} [i^2 + 2i + 1]c_i x^i - \sum_{i=0}^{\infty} i^2 c_i x^{i-1} = 0$$

or

$$\sum_{i=0}^{\infty} (i+1)^2 c_i x^i - \sum_{i=-1}^{\infty} (i+1)^2 c_{i+1} x^i = 0$$

The lowest, $i = -1$ coefficient is zero automatically, thus c_0 is arbitrary. The remaining coefficients are

$$(i+1)^2[c_i - c_{i+1}]$$

set to zero $\Rightarrow c_{i+1} = c_i$ for $i = 0, 1, 2,.... \Rightarrow c_0 = c_1 = c_2 = c_3 = ... \Rightarrow 1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$ is the first basic solution. Again, we can get the second basic solution by either the V of P or Frobenius technique. We demonstrate only the latter:

$$y^{(T)} = \frac{\ln x}{1-x} + \sum_{i=0}^{\infty} c_i^* x^{i+0}$$

getting the same left hand side and the following right hand side:

$$x(x-1)\left[\frac{2}{x(1-x)^2} - \frac{1}{x^2(1-x)}\right]$$
$$+ (3x-1) \cdot \frac{1}{x(1-x)} = 0$$

[not typical, but it may happen]. This means that not only c_0^*, but all the other c^*-coefficients can be set equal to zero. The second basic solution is thus $\frac{\ln x}{1-x}$ [which can be verified easily by direct substitution].

$r_1 - r_2$ **Equals a Positive Integer**

(we *choose* $r_1 > r_2$).

The first basic solution can be constructed, based on $y^{(T)} = \sum\limits_{i=0}^{\infty} c_i x^{i+r_1}$, in the usual manner (don't forget that r_1 should be the *bigger* root). The second basic solution will then have the form of

$$Ky_1 \ln x + \sum_{i=0}^{\infty} c_i^* x^{i+r_2}$$

where K becomes one of the *unknowns* (on par with the c_i^*'s), but it may turn out to have a zero value. Note that we will first have a free choice of c_0^* (must be non-zero) and then, when we reach it, we will also be offered a free choice of $c_{r_1-r_2}^*$ (to simplify the solution, we usually set it equal to zero – a nonzero choice would only add an extra multiple of y_1).

EXAMPLES:

(1)

$$(x^2-1)x^2y'' - (x^2+1)xy' + (x^2+1)y = 0$$

$[a(x) = -\frac{x^2+1}{x^2-1}$ and $b(x) = \frac{x^2+1}{x^2-1}] \Rightarrow$

$$r^2 - 1 = 0$$

$\Rightarrow r_{1,2} = 1$ and -1. Using $y^{(T)} = \sum\limits_{i=0}^{\infty} c_i x^{i+1}$ we get:

$$\sum_{i=0}^{\infty}(i+1)ic_i x^{i+3} - \sum_{i=0}^{\infty}(i+1)ic_i x^{i+1} - \sum_{i=0}^{\infty}(i+1)c_i x^{i+3}$$
$$- \sum_{i=0}^{\infty} c_i(i+1)x^{i+1} + \sum_{i=0}^{\infty} c_i x^{i+3} + \sum_{i=0}^{\infty} c_i x^{i+1} = 0$$

$\Leftrightarrow$

$$\sum_{i=0}^{\infty} i^2 c_i x^{i+3} - \sum_{i=0}^{\infty} i(i+2)c_i x^{i+1} = 0$$

$\Leftrightarrow$

$$\sum_{i=0}^{\infty} i^2 c_i x^{i+3} - \sum_{i=-2}^{\infty} (i+2)(i+4)c_{i+2}x^{i+3} = 0$$

The lowest $i = -2$ term is zero automatically [$\Rightarrow c_0$ can have any value], the next $i = -1$ term [still 'exceptional'] disappears only when $c_1 = 0$. The rest of the c-sequence follows from

$$c_{i+2} = \frac{i^2 c_i}{(i+2)(i+4)}$$

with $i = 0, 1, 2, ... \Rightarrow c_2 = c_3 = c_4 = ... = 0$. The first basic solution is thus $c_0 x$ [$y_1 = x$, discarding the constant]. To construct the second basic solution, we substitute

$$Kx \ln x + \sum_{i=0}^{\infty} c_i^* x^{i-1}$$

for y, getting:

$$\sum_{i=0}^{\infty}(i-1)(i-2)c_i x^{i+1} - \sum_{i=0}^{\infty}(i-1)(i-2)c_i x^{i-1}$$
$$-\sum_{i=0}^{\infty}(i-1)c_i x^{i+1} - \sum_{i=0}^{\infty} c(i-1)_i x^{i-1}$$
$$+\sum_{i=0}^{\infty} c_i x^{i+1} + \sum_{i=0}^{\infty} c_i x^{i-1} =$$
$$\sum_{i=0}^{\infty}(i-2)^2 c_i x^{i+1} - \sum_{i=0}^{\infty} i(i-2)c_i x^{i-1} =$$
$$\sum_{i=0}^{\infty}(i-2)^2 c_i x^{i+1} - \sum_{i=-2}^{\infty}(i+2)i c_{i+2} x^{i+1}$$

on the left hand side, and

$$-(x^2-1)x^2 \cdot \frac{K}{x} + (x^2+1)x \cdot K = 2Kx$$

on the right hand side (the contribution of $Kx \ln x$). The $i = -2$ term allows c_0^* to be arbitrary, $i = -1$ requires $c_1^* = 0$, and $i = 0$ [due to the right hand side, the x^1-terms must be also considered 'exceptional'] requires

$$4c_0^* = 2K$$

$\Rightarrow K = 2c_0^*$, and leaves c_2^* free for us to choose (we take $c_2^* = 0$). After that,

$$c_{i+2}^* = \frac{(i-2)^2}{(i+2)i} c_i^*$$

where $i = 1, 2, 3, \; .... \Rightarrow c_3^* = c_4^* = c_5^* = ... = 0$. The second basic solution is thus

$$c_0^* \left(2x \ln x + \frac{1}{x} \right)$$

[verify!].

(2)

$$x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0$$

$\Rightarrow r^2 - \frac{1}{4} = 0 \Rightarrow r_{1,2} = \frac{1}{2}$ and $-\frac{1}{2}$. Substituting

$$y^{(T)} = \sum_{i=0}^{\infty} c_i x^{i+1/2}$$

we get

$$\begin{aligned}
\sum_{i=0}^{\infty} (i+\frac{1}{2})(i-\frac{1}{2}) c_i x^{i+1/2} + \sum_{i=0}^{\infty} (i+\frac{1}{2}) c_i x^{i+1/2} & \\
+ \sum_{i=0}^{\infty} c_i x^{i+5/2} - \frac{1}{4} \sum_{i=0}^{\infty} c_i x^{i+1/2} &= 0 \\
\sum_{i=0}^{\infty} (i+1) i c_i x^{i+1/2} + \sum_{i=0}^{\infty} c_i x^{i+5/2} &= 0 \\
\sum_{i=-2}^{\infty} (i+3)(i+2) c_{i+2} x^{i+5/2} + \sum_{i=0}^{\infty} c_i x^{i+5/2} &= 0
\end{aligned}$$

which yields: a free choice of c_0, $c_1 = 0$ and

$$c_{i+2} = -\frac{c_i}{(i+2)(i+3)}$$

where $i = 0,\ 1, ... \Rightarrow$

$$\begin{aligned} c_2 &= -\frac{c_0}{3!} \\ c_4 &= \frac{c_0}{5!} \\ c_6 &= -\frac{c_0}{7!} \\ &\ldots \end{aligned}$$

and $c_3 = c_5 = ... = 0$. The first basic solution thus equals

$$\begin{aligned} & c_0 x^{1/2}(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...) \\ = \ & c_0 \frac{\sin x}{\sqrt{x}} \end{aligned}$$

$[\Rightarrow y_1 = \frac{\sin x}{\sqrt{x}}]$. Substituting

$$K y_1 \ln x + \sum_{i=0}^{\infty} c_i^* x^{i-1/2}$$

for y similarly reduces the equation to

$$\sum_{i=0}^{\infty} (i-1) i c_i^* x^{i-1/2} + \sum_{i=0}^{\infty} c_i^* x^{i+3/2}$$

on the left hand side and

$$\begin{aligned} & -x^2 \cdot (-\frac{y_1}{x^2} + \frac{2}{x} y_1') - \ x \cdot \frac{y_1}{x} = \\ & -2xy_1' = \ K(-x^{1/2} + \ \frac{5x^{5/2}}{3!} - \frac{9x^{9/2}}{5!} + ...) \end{aligned}$$

on the right hand side or, equivalently,

$$\begin{aligned} & \sum_{i=-2}^{\infty} (i+1)(i+2) c_{i+2}^* x^{i+3/2} + \sum_{i=0}^{\infty} c_i^* x^{i+3/2} = \\ & K(-x^{1/2} + \frac{5x^{5/2}}{3!} - \frac{9x^{9/2}}{5!} + ...) \end{aligned}$$

This implies that c_0^* can have any value ($i = -2$), c_1^* can also have any value (we *make* it 0), K *must* equal zero ($i = -1$), and

$$c_{i+2}^* = -\frac{c_i^*}{(i+1)(i+2)}$$

for $i = 0, 1, 2, \ldots \Rightarrow$

$$\begin{aligned} c_2^* &= -\frac{c_0^*}{2!} \\ c_4^* &= \frac{c_0^*}{4!} \\ c_6^* &= -\frac{c_0^*}{6!} \\ &\cdots \end{aligned}$$

$\Rightarrow$

$$y_2 = x^{-1/2}(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ....) = \frac{\cos x}{\sqrt{x}}$$

In each of the previous examples the second basic solution could have been constructed by V of P – try it.

Also note that so far we have avoided solving a truly non-homogeneous recurrence formula – K never appeared in more than one of its (infinitely many) equations.

A few Special functions of Mathematical Physics

This section demonstrates various applications of the Frobenius technique.

Laguerre Equation

$$y'' + \frac{1-x}{x}y' + \frac{n}{x}y = 0$$

or, equivalently:

$$(xe^{-x}y')' + nye^{-x} = 0$$

which identifies it as an eigenvalue problem, with the solutions being orthogonal in the $\int_0^\infty e^{-x}L_{n_1}(x) \cdot L_{n_2}(x)\, dx$ sense.

Since $a(x) = 1 - x$ and $b(x) = x$ we get $r^2 = 0$ [duplicate roots]. Substituting $\sum_{i=0}^{\infty} c_i x^i$ for y in the original equation (multiplied by x) results in

$$\sum_{i=0}^{\infty} i^2 c_i x^{i-2} + \sum_{i=0}^{\infty} (n-i) c_i x^{i-1} = 0$$

$\Leftrightarrow$

$$\sum_{i=-1}^{\infty} (i+1)^2 c_{i+1} x^{i-1} + \sum_{i=0}^{\infty} (n-i) c_i x^{i-1} = 0$$

$\Rightarrow$

$$c_{i+1} = -\frac{n-i}{(i+1)^2} c_i$$

for $i = 0,\ 1,\ 2,\ ....$ Only polynomial solutions are square integrable in the above sense (relevant to Physics), so n must be an *integer*, to make c_{n+1} and all subsequent c_i-values equal to 0 and thus solve the eigenvalue problem.

The first basic solution is thus $L_n(x)$ [the standard notation for Laguerre polynomials]:

$$1 - \frac{n}{1^2} x + \frac{n(n-1)}{(2!)^2} x^2 - \frac{n(n-1)(n-2)}{(3!)^2} x^3 + ... \pm \frac{1}{n!} x^n$$

The second basic solution does *not* solve the eigenvalue problem (it is not square integrable), so we will not bother to construct it [not that it should be difficult – try it if you like].

Bessel equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

where n has any (non-negative) value.

The **indicial equation** is $r^2 - n^2 = 0$ yielding $r_{1,2} = n,\ -n$.

To build the **first basic solution** we use

$$y^{(T)} = \sum_{i=0}^{\infty} c_i x^{i+n}$$

$\Rightarrow$

$$\sum_{i=0}^{\infty} i(i+2n) c_i x^{i+n} + \sum_{i=0}^{\infty} c_i x^{i+n+2} = 0$$

$\Leftrightarrow$

$$\sum_{i=0}^{\infty} i(i+2n)c_i x^{i+n} + \sum_{i=2}^{\infty} c_{i-2}x^{i+n} = 0$$

$\Rightarrow c_0$ arbitrary, $c_1 = c_3 = c_5 = ... = 0$ and

$$c_i = -\frac{c_{i-2}}{i(2n+i)}$$

for $i = 2,\ 4,\ 6,\ ... \Rightarrow$

$$\begin{aligned}
c_2 &= -\frac{c_0}{2(2n+2)} \\
c_4 &= \frac{c_0}{4 \times 2 \times (2n+2) \times (2n+4)} \\
c_6 &= -\frac{c_0}{6 \times 4 \times 2 \times (2n+2) \times (2n+4) \times (2n+6)} \\
&\cdots \\
c_{2k} &= \frac{(-1)^k c_0}{2^{2k}(n+1)(n+2).....(n+k)k!}
\end{aligned}$$

in general, where $k = 0,\ 1,\ 2,\ ....$ When n is an *integer*, the last expression can be written as

$$c_{2k} = \frac{(-1)^k n! c_0}{2^{2k}(n+k)!k!} \equiv \frac{(-1)^k \tilde{c}_o}{2^{2k+n}\, k!\,(n+k)!}$$

The first basic solution is thus

$$\sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+n}}{k!(n+k)!}$$

It is called the Bessel function of the *first kind* of 'order' n [note that the 'order' has nothing to do with the order of the corresponding equation, which is always 2], the standard *notation* being $J_n(x)$; its values (if not on your calculator) can be found in tables.

When n is a **non-integer**, one has to extend the definition of the factorial function to non-integer arguments. This extension is called a Γ-function, and is 'shifted' with respect to the factorial function, thus: $n! \equiv \Gamma(n+1)$. For positive α $(= n+1)$ values, it is achieved by the following integral

$$\Gamma(\alpha) \equiv \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

[note that for integer α this yields $(\alpha-1)!$], for negative α values the extension is done with the help of

$$\Gamma(\alpha-1)=\frac{\Gamma(\alpha)}{\alpha-1}$$

[its values can often be found on your calculator].

Using this extension, the previous $J_n(x)$ solution (of the Bessel equation) becomes correct for any n [upon the $(n+k)!\to\Gamma(n+k+1)$ replacement].

When n is *not* an integer, the same formula with $n\to-n$ provides *the second basic solution* [easy to verify].

Of the non-integer cases, the most important are those with a *half-integer* value of n. One can easily verify [you will need $\Gamma(\frac{1}{2})=\sqrt{\pi}$] that the corresponding Bessel functions are elementary, e.g.

$$\begin{aligned}
J_{\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\sin x \\
J_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\cos x \\
J_{\frac{3}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right) \\
&\cdots
\end{aligned}$$

Unfortunately, the most common is the case of n being an *integer*.

Constructing the *second basic solution* is then a lot more difficult. It has, as we know, the form of

$$Ky_1\ln x+\sum_{i=0}^{\infty}c_i^*x^{i-n}$$

Substituting this into the Bessel equation yields

$$\sum_{i=0}^{\infty}i(i-2n)c_i^*x^{i-n}+\sum_{i=2}^{\infty}c_{i-2}^*x^{i-n}$$

on the left hand side and

$$
\begin{aligned}
-K\left[x^2\cdot(2\frac{y_1'}{x}-\frac{y_1}{x^2})+x\cdot\frac{y_1}{x}\right] &= \\
-2K\sum_{k=0}^{\infty}\frac{(-1)^k(2k+n)(\frac{x}{2})^{2k+n}}{k!(n+k)!} &= \\
-2K\sum_{k=n}^{\infty}\frac{(-1)^{k-n}(2k-n)(\frac{x}{2})^{2k-n}}{(k-n)!k!}
\end{aligned}
$$

on the right hand side of the recurrence formula.

One can solve it by taking c_0^* to be arbitrary, $c_1^*=c_3^*=c_5^*=....=0$, and

$$
\begin{aligned}
c_2^* &= \frac{c_0^*}{2(2n-2)} \\
c_4^* &= \frac{c_0^*}{4\times 2\times(2n-2)\times(2n-4)} \\
c_6^* &= \frac{c_0^*}{6\times 4\times 2\times(2n-2)\times(2n-4)\times(2n-6)} \\
& ....
\end{aligned}
$$

$$
\begin{aligned}
c_{2k}^* &= \frac{c_0^*}{2^{2k}(n-1)(n-2)....(n-k)k!} \\
&\equiv \frac{c_0^*(n-k-1)!}{2^{2k}(n-1)!k!}
\end{aligned}
$$

up to and including $k=n-1$ $[i=2n-2]$. When we reach $i=2n$ the right hand side starts contributing! The overall coefficient of x^n is

$$c_{2n-2}^*=-2K\frac{1}{2^n\,(n-1)!}$$

$\Rightarrow$

$$K=\frac{-c_0^*}{2^{n-1}(n-1)!}$$

allowing a free choice of c_{2n}^*.

To solve the remaining part of the recurrence formula (truly non-homogeneous) is more difficult, so we only quote (and verify) the answer:

$$c_{2k}^*=c_0^*\frac{(-1)^{k-n}(h_{k-n}+h_k)}{2^{2k}(k-n)!k!(n-1)!}$$

for $k \geq n$, where $h_k = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k}$.

The second basic solution is usually written (a slightly different normalizing constant is used, and a bit of $J_n(x)$ is added) as:

$$\begin{aligned} Y_n(x) &= \frac{2}{\pi} J_n(x) \left[\ln \frac{x}{2} + \gamma \right] \\ &+ \frac{1}{\pi} \sum_{k=n}^{\infty} \frac{(-1)^{k+1-n}(h_{k-n} + h_k)}{(k-n)!k!} \left(\frac{x}{2} \right)^{2k-n} \\ &- \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2} \right)^{2k-n} \end{aligned}$$

where γ is the Euler constant ≈ 0.557 [the reason for the extra term is that the last formula is derived based on yet another, possibly more elegant approach than ours, namely:

$$\lim_{\upsilon \to n} \frac{J_\nu \cos(\nu\pi) - J_{-\nu}}{\sin(\nu\pi)}$$

$Y_n(x)$ is called the Bessel function of *second kind* of order n.

Modified Bessel equation:

$$x^2 y'' + xy' - (x^2 + n^2) y = 0$$

[differs from Bessel equation by a single sign]. The two basic solutions can be developed in almost an identical manner to the 'unmodified' Bessel case [the results differ only be an occasional sign]. We will not duplicate our effort, and only mention the new notation: the two basic solutions are now $I_n(x)$ and $K_n(x)$ [modified Bessel functions of first and second kind]. Only I_0 and I_1 need to be tabulated as $I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x} I_n$ and $I_n' = \frac{I_{n-1}+I_{n+1}}{2}$ (same with $I_n \to K_n$).

Transformed Bessel equation:

$$x^2 y'' + (1-2a)xy' + (b^2 c^2 x^{2c} - n^2 c^2 + a^2) y = 0$$

where a, b, c and n are arbitrary constants [the equation could have been written as

$$x^2 y'' + Axy' + (B^2 x^C - D) y = 0$$

but the above parametrization is more convenient].

To find the solution we substitute $y(x) = x^a \cdot u(x)$ [introducing new *dependent* variable u] getting:

$$a(a-1)u + 2axu' + x^2u'' + (1-2a)(au + xu') \\ + (b^2c^2x^{2c} - n^2c^2 + a^2)u =$$

$$x^2u'' + xu' + (b^2c^2x^{2c} - n^2c^2)u = 0$$

Then we introduce $z = bx^c$ as a new *independent* variable [recall that

$$u' \to \frac{du}{dz} \cdot bcx^{c-1}$$

and

$$u'' \to \frac{d^2u}{dz^2} \cdot (bcx^{c-1})^2 + \frac{du}{dz} \cdot bc(c-1)x^{c-2}$$

] $\Rightarrow$

$$x^2 \cdot \left(\frac{d^2u}{dz^2} \cdot (bcx^{c-1})^2 + \frac{du}{dz} \cdot bc(c-1)x^{c-2} \right) + \\ x \cdot \left(\frac{du}{dz} \cdot bcx^{c-1} \right) + (b^2c^2x^{2c} - n^2c^2)u =$$

[after cancelling c^2]

$$z^2 \cdot \frac{d^2u}{dz^2} + z \cdot \frac{du}{dz} + \left(z^2 - n^2\right) u = 0$$

which is the Bessel equation, having

$$u(z) = C_1 J_n(z) + C_2 Y_n(z)$$

[or $C_2 J_{-n}(x)$ when n is *not* an integer] as its general solution.

The solution to the *original* equation is thus

$$C_1 x^a J_n(bx^c) + C_2 x^a Y_n(bx^c)$$

EXAMPLES:

1.

$$xy'' - y' + xy = 0$$

[same as $x^2y'' - xy' + x^2y = 0$] $\Rightarrow a = 1$ [from $1 - 2a = -1$], $c = 1$ [from $b^2c^2x^{2c}y = x^2y$], $b = 1$ [from $b^2c^2 = 1$] and $n = 1$ [from $a^2 - n^2c^2 = 0$] $\Rightarrow$

$$y(x) = C_1 x J_1(x) + C_2 x Y_1(x)$$

2.

$$x^2y'' - 3xy' + 4(x^4 - 3)y = 0$$

$\Rightarrow a = 2$ [from $1 - 2a = -3$], $c = 2$ [from $b^2c^2x^{2c}y = 4x^4y$], $b = 1$ [from $b^2c^2 = 4$] and $n = 2$ [from $a^2 - n^2c^2 = -12$] $\Rightarrow$

$$y = C_1 x^2 J_2(x^2) + C_2 x^2 Y_2(x^2)$$

3.

$$x^2y'' + (\frac{81}{4}x^3 - \frac{35}{4})y = 0$$

$\Rightarrow a = \frac{1}{2}$ [from $1 - 2a = 0$], $c = \frac{3}{2}$ [from x^3], $b = 3$ [from $b^2c^2 = \frac{81}{4}$] and $n = 2$ [from $a^2 - n^2c^2 = -\frac{35}{4}$] $\Rightarrow$

$$y = C_1\sqrt{x}J_2(3x^{3/2}) + C_2\sqrt{x}Y_2(3x^{3/2})$$

4.

$$x^2y'' - 5xy' + (x + \frac{35}{4})y = 0$$

$\Rightarrow a = 3$ [$1 - 2a = -5$], $c = \frac{1}{2}$ [xy], $b = 2$ [$b^2c^2 = 1$] and $n = 1$ [$a^2 - n^2c^2 = \frac{35}{4}$] $\Rightarrow$

$$y = C_1 x^3 J_1(2\sqrt{x}) + C_2 x^3 Y_1(2\sqrt{x})$$

Hypergeometric equation

$$x(1 - x)y'' + [c - (a + b + 1)x]y' - aby = 0$$

$\Rightarrow r^2 + r(c - 1) = 0 \Rightarrow r_{1,2} = 0$ and $1 - c$.

Substituting $y^{(T)} = \sum_{i=0}^{\infty} c_i x^i$ yields:

$$\sum_{i=-1}^{\infty}(i + 1)(i + c)c_{i+1}x^i - \sum_{i=0}^{\infty}(i + a)(i + b)c_i x^i$$

$\Rightarrow$

$$\begin{aligned}
c_1 &= \frac{ab}{1 \cdot c} c_0 \\
c_2 &= \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} c_0 \\
c_3 &= \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} c_0 \\
& \quad ....
\end{aligned}$$

which shows that the *first basic solution* is

$$\begin{aligned}
& 1 + \frac{ab}{1 \cdot c} x + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} x^2 \\
& + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} x^3 + \; ....
\end{aligned}$$

The usual *notation* for this series is $F(a,b;c;x)$, and it is called the hypergeometric function. Note that a and b are interchangeable.. Also note that when either of them is a negative integer (or zero), $F(a,b;c;x)$ is just a simple polynomial (of the corresponding degree) – please learn to identify it as such!

Similarly, when c is *noninteger* [to avoid Case 3], we can show [skipping the details now] that the *second basic solution* is

$$x^{1-c} F(a+1-c, b+1-c; 2-c; x)$$

[this may be correct even in some Case 3 situations, but don't forget to verify it].

EXAMPLE:

$$x(1-x)y'' + (3-5x)y' - 4y = 0$$

$\Rightarrow ab = 4$, $a+b+1 = 5 \Rightarrow b^2 - 4b + 4 = 0 \Rightarrow a = 2$, $b = 2$, and $c = 3 \Rightarrow$

$$C_1 F(2,2;3;x) + C_2 x^{-2} F(0,0,;-1;x)$$

[the second part is subject to verification]. Since $F(0,0;-1;x) \equiv 1$, the second basic solution is x^{-2}, which *does meet* the equation [substitute].

Transformed Hypergeometric equation:

$$(x - x_1)(x_2 - x)y'' + [D - (a + b + 1)x]y' - aby = 0$$

where x_1 and x_2 (in addition to a, b, and D) are specific *numbers*.

One can easily verify that changing the *independent* variable to $z = \dfrac{x - x_1}{x_2 - x_1}$ transforms the equation to

$$z(1-z)\frac{d^2y}{dz^2} + \left[\frac{D - (a + b + 1)x_1}{x_2 - x_1} - (a + b + 1)z\right]\frac{dy}{dz}$$
$$- aby = 0$$

which we know how to solve [hypergeometric].

EXAMPLES:

1.

$$4(x^2 - 3x + 2)y'' - 2y' + y = 0$$

$\Rightarrow$

$$(x - 1)(2 - x)y'' + \frac{1}{2}y' - \frac{1}{4}y = 0$$

$\Rightarrow x_1 = 1$, $x_2 = 2$, $ab = \frac{1}{4}$ and $a + b + 1 = 0 \Rightarrow b^2 + b + \frac{1}{4} = 0 \Rightarrow a = -\frac{1}{2}$ and $b = -\frac{1}{2}$, and finally $c = \frac{\frac{1}{2} - (a+b+1)x_1}{x_2 - x1} = \frac{1}{2}$. The solution is thus

$$y = C_1 F(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; x - 1) + C_2(x - 1)^{1/2}F(0, 0; \frac{3}{2}; x - 1)$$

[since $z = x - 1$]. Note that

$$F(0, 0; \frac{3}{2}; x - 1) \equiv 1$$

[*some* hypergeometric functions are elementary or even trivial, e.g. $F(1, 1; 2; x) \equiv -\frac{\ln(1-x)}{x}$, etc.].

2.

$$3x(1 + x)y'' + xy' - y = 0$$

$\Rightarrow$

$$(x+1)\,(0-x)\,y'' - \frac{1}{3}xy + \frac{1}{3}y = 0$$

$\Rightarrow x_1 = -1$ [note the sign!] $x_2 = 0$, $ab = -\frac{1}{3}$ and $a+b+1 = \frac{1}{3} \Rightarrow a = \frac{1}{3}$ and $b = -1 \Rightarrow c = \frac{0-\frac{1}{3}(-1)}{1} = \frac{1}{3} \Rightarrow$

$$y = C_1 F(\frac{1}{3}, -1; \frac{1}{3}; x+1) + C_2 (x+1)^{2/3} F(1, -\frac{1}{3}; \frac{5}{3}; x+1)$$

[the first $F(...)$ equals to $-x$; coincidentally, even the second $F(...)$ can be converted to a rather lengthy expression involving ordinary functions].