Solve

$$x^2y'' + \frac{1}{4}(x + \frac{3}{4})y = 0$$

Hint:

$$\begin{aligned} y(x) &= \sqrt{x} \cdot u(x) \\ \sqrt{x} &= z \end{aligned}$$

Let's start with

$$\begin{aligned} y &\rightarrow x^{1/2}u \\ y' &\rightarrow \tfrac{1}{2}x^{-1/2}u + x^{1/2}u' \\ y'' &\rightarrow -\tfrac{1}{4}x^{-3/2}u + x^{-1/2}u' + x^{1/2}u'' \end{aligned}$$

Substituting *and* dividing by $x^{1/2}$ yields

$$\begin{aligned} x^2u'' + xu' - \tfrac{1}{4}u + \tfrac{1}{4}(x + \tfrac{3}{4})u &= \\ x^2u'' + xu' + \tfrac{1}{4}xu - \tfrac{1}{16}u &= 0 \end{aligned}$$

Secondly

$$\begin{aligned} \frac{du}{dx} &\rightarrow \frac{du}{dz} \cdot \tfrac{1}{2}x^{-1/2} \\ \frac{d^2u}{dx^2} &\rightarrow \frac{d^2u}{dz^2} \cdot \tfrac{1}{4}x^{-1} - \frac{du}{dz} \cdot \tfrac{1}{4}x^{-3/2} \end{aligned}$$

results in

$$\begin{aligned} \frac{d^2u}{dz^2} \cdot \tfrac{1}{4}x - \frac{du}{dz} \cdot \tfrac{1}{4}x^{1/2} + \frac{du}{dz} \cdot \tfrac{1}{2}x^{1/2} + \tfrac{1}{4}xu - \tfrac{1}{16}u &= \\ \frac{d^2u}{dz^2} \cdot \tfrac{1}{4}x + \frac{du}{dz} \cdot \tfrac{1}{4}x^{1/2} + \tfrac{1}{4}xu - \tfrac{1}{16}u &= 0 \end{aligned}$$

Finally, we have to replace x by z^2 (we also multiply the whole equation by 4):

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + z^2u - \tfrac{1}{4}u = 0$$

which is (regular) Bessel equation with $n = \frac{1}{2}$.

Solution:

$$u(z) = A \cdot J_{1/2}(z) + B \cdot J_{-1/2}(z)$$

or

$$u(x) = A \cdot J_{1/2}(\sqrt{x}) + B \cdot J_{-1/2}(\sqrt{x})$$

which implies that

$$y(x) = A \cdot \sqrt{x} \cdot J_{1/2}(\sqrt{x}) + B \cdot \sqrt{x} \cdot J_{-1/2}(\sqrt{x})$$