MATH2F05 FIRST MIDTERM OCTOBER 23, 2003

Find all solutions to:

1. $\left(y - 2\frac{x}{y}\right) dx + \left(2x + \frac{1}{y}\right) dy = 0$

Hint: Find the integrating factor first.

$$\begin{aligned}
\frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} &= \frac{1}{y} \\
\ln F &= \int \frac{dy}{y} = \ln y \\
F &= y \\
\left(y^2 - 2x\right) dx + (2xy + 1)\, dy &= 0 \\
G &= xy^2 - x^2 \\
H &= 2xy + 1 - \frac{\partial G}{\partial y} = 1 \\
G + \int H\ dy &= xy^2 - x^2 + y = C \\
y &= \frac{-1 \pm \sqrt{1 + 4x^3 + 4Cx}}{2x}
\end{aligned}$$

2. $2xyy' + y^2 = 2x + 1$

$$y' + \frac{y}{2x} = \left(1 + \frac{1}{2x}\right) y^{-1}$$

Bernoulli, $a = -1$, $1 - a = 2$, $u = y^2$

$$u' + \frac{u}{x} = 2 + \frac{1}{x}$$

Hmogenoues:

$$\begin{aligned}
\frac{du}{u} &= -\frac{dx}{x} \\
u &= \frac{c}{x} \\
\frac{c'}{x} &= 2 + \frac{1}{x} \\
c &= x^2 + x + C \\
u &= x + 1 + \frac{C}{x} \\
y &= \pm\sqrt{x + 1 + \frac{C}{x}}
\end{aligned}$$

3. $y = xy' + \dfrac{1}{1+y'}$ (3 out of 4 marks given for the *singular* solution).

$$y = Cx + \frac{1}{1+C}$$

(regular family of solutions).

$$\begin{aligned}
x &= -g'(p) = \frac{1}{(1+p)^2} \\
p &= \pm\frac{1}{\sqrt{x}} - 1 \\
y &= \pm\sqrt{x} - x \pm \sqrt{x} = \pm 2\sqrt{x} - x
\end{aligned}$$

(singular solution).

4. $(x-1)y'' - xy' + y = 0$ Hint: $y = x$ is a solution.

$$\begin{aligned}
y_{\mathrm{t}} &= c \cdot x \\
y_{\mathrm{t}}' &= c' \cdot x + c \\
y_{\mathrm{t}}'' &= c'' \cdot x + 2c
\end{aligned}$$

Substitute:

$$(x-1)(c''x + 2c') - x^2c' = x(x-1)c'' - (x^2 - 2x + 2)c' = 0$$

Introduce $z = c'$

$$\begin{aligned}
\frac{dz}{z} &= \frac{x^2 - 2x + 2}{x(x-1)} = 1 - \frac{2}{x} + \frac{1}{x-1} \\
\ln z &= x - 2\ln x + \ln(x-1) \\
z &= \frac{x-1}{x^2}\ e^x \\
c &= \int \frac{x-1}{x^2}\ e^x\ dx = \frac{e^x}{x} \\
y_2 &= \frac{e^x}{x} \cdot x = e^x \\
y &= C_1 x + C_2 e^x
\end{aligned}$$

5. $4y'' - 12y' + 9y = 0$ subject to $y(0) = 2$ and $y'(0) = -3$.

$$\begin{aligned}
\lambda^2 - 3\lambda + \frac{9}{4} &= 0 \\
\lambda_{1,2} &= \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{9}{4}} = \frac{3}{2} \pm 0 \\
y &= C_1 \exp(\frac{3}{2}x) + C_2 x \exp(\frac{3}{2}x) \\
2 &= C_1 \\
-3 &= \frac{3}{2}C_1 + C_2 \\
C_2 &= -6 \\
y &= 2(1 - 3x)\exp(\frac{3}{2}x)
\end{aligned}$$