The case of **Double (Multiple) Eigenvalue**

For each such eigenvalue we must first find *all* possible solutions of the

$$\mathbf{q}e^{\lambda x}$$

type (i.e. find all LI eigenvectors), then (if we get fewer eigenvectors than the multiplicity of λ) we have to find all possible solutions having the form of

$$(\mathbf{q}x + \mathbf{s})e^{\lambda x}$$

where $\mathbf{q}$ and $\mathbf{s}$ are two constant vectors to be found by substituting this (trial) solution into the basic equation $\mathbf{y}' = \mathbb{A}\mathbf{y}$**.** As a result we get

$$\mathbf{q} = (\mathbb{A} - \lambda\mathbb{I})\mathbf{q}\,x + (\mathbb{A} - \lambda\mathbb{I})\mathbf{s}$$

This further implies that $\mathbf{s}$ must be a solution to

$$(\mathbb{A} - \lambda\mathbb{I})^2\mathbf{s} = \mathbf{0}$$

And, if still not done, we have to proceed to

$$(\mathbf{q}\frac{x^2}{2!} + \mathbf{s}x + \mathbf{u})e^{\lambda x}$$

which implies

$$\begin{aligned} (\mathbb{A} - \lambda\mathbb{I})\mathbf{q} &= \mathbf{0} \\ \mathbf{q} &= (\mathbb{A} - \lambda\mathbb{I})\mathbf{s} \\ \mathbf{s} &= (\mathbb{A} - \lambda\mathbb{I})\mathbf{u} \end{aligned}$$

where, clearly, $\mathbf{u}$ is a solution to

$$(\mathbb{A} - \lambda\mathbb{I})^3\mathbf{u} = \mathbf{0}$$

etc.

EXAMPLES:

1.

$$\mathbf{A} = \begin{array}{|c|c|c|}\hline 5 & 2 & 2 \\ \hline 2 & 2 & -4 \\ \hline 2 & -4 & 2 \\ \hline \end{array}$$

has $\lambda^3 - 9\lambda^2 + 108$ as its characteristic polynomial [hint: there is a double root] $\Rightarrow 3\lambda^2 - 18\lambda = 0$ has two roots, 0 [does not check] and 6 [checks]. Furthermore, $(\lambda^3 - 9\lambda^2 + 108) \div (\lambda - 6)^2 = \lambda + 3 \Rightarrow$ the three eigenvalues are -3 and 6 [duplicate].

Using $\lambda = -3$ we get:

$$\begin{array}{|c|c|c||}\hline 8 & 2 & 2 \\ \hline 2 & 5 & -4 \\ \hline 2 & -4 & 5 \\ \hline \end{array} \overset{GJ}{\Rightarrow} \begin{array}{|c|c|c||}\hline 1 & 0 & \frac{1}{2} \\ \hline 0 & 1 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \Rightarrow \mathbf{q} = \begin{array}{|c|}\hline -1 \\ \hline 2 \\ \hline 2 \\ \hline \end{array}$$

which, when multiplied by e^{-3x}, gives the first basic solution.

Using $\lambda = 6$ yields:

$$\begin{array}{|c|c|c||}\hline -1 & 2 & 2 \\ \hline 2 & -4 & -4 \\ \hline 2 & -4 & -4 \\ \hline \end{array} \overset{GJ}{\Rightarrow} \begin{array}{|c|c|c||}\hline 1 & -2 & -2 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \Rightarrow \mathbf{q} = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 0 \\ \hline \end{array} \text{ and } \begin{array}{|c|}\hline 2 \\ \hline 0 \\ \hline 1 \\ \hline \end{array}$$

which, when multiplied by e^{6x}, supplies the remaining two basic solutions.

2.

$$\mathbb{A} = \begin{array}{|c|c|c|}\hline 1 & -3 & 1 \\ \hline 2 & -1 & -2 \\ \hline 2 & -3 & 0 \\ \hline \end{array}$$

has $\lambda^3 - 3\lambda - 2$ as its characteristic polynomial, with roots: $\lambda_1 = -1$ [one of our rules] $\Rightarrow (\lambda^3 - 3\lambda - 2) \div (\lambda + 1) = \lambda^2 - \lambda - 2 \Rightarrow \lambda_2 = -1$ and $\lambda_3 = 2$. So again, there is one duplicate root.

For $\lambda = 2$ we get:

$$\begin{array}{|c|c|c||}\hline -1 & -3 & 1 \\ \hline 2 & -3 & -2 \\ \hline 2 & -3 & -2 \\ \hline \end{array} \overset{GJ}{\Rightarrow} \begin{array}{|c|c|c||}\hline 1 & 0 & -1 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \Rightarrow \mathbf{y}_{(1)} = \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline 1 \\ \hline \end{array} e^{2x}$$

For $\lambda = -1$ we get:

$$\begin{array}{|c|c|c||}\hline 2 & -3 & 1 \\ \hline 2 & 0 & -2 \\ \hline 2 & -3 & 1 \\ \hline \end{array} \overset{GJ}{\Rightarrow} \begin{array}{|c|c|c||}\hline 1 & 0 & -1 \\ \hline 0 & 1 & -1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \Rightarrow \mathbf{y}_{(2)} = \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array} e^{-x}$$

[a *single* solution only]. The challenge is to construct the other (last) solution. Squaring the previous matrix and adding the extra row yields

$$\begin{array}{|c|c|c||}\hline 0 & -9 & 9 \\ \hline 0 & 0 & 0 \\ \hline 0 & -9 & 9 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \overset{GJ}{\Rightarrow} \begin{array}{|c|c|c||}\hline 1 & 0 & 2 \\ \hline 0 & 1 & -1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

whose solution is

$$\mathbf{s} = \begin{array}{|c|}\hline -2 \\ \hline 1 \\ \hline 1 \\ \hline \end{array}$$

Since

$$\mathbf{q}_s = \begin{array}{|c|c|c||}\hline 2 & -3 & 1 \\ \hline 2 & 0 & -2 \\ \hline 2 & -3 & 1 \\ \hline \end{array} \cdot \begin{array}{|c|}\hline -2 \\ \hline 1 \\ \hline 1 \\ \hline \end{array} = \begin{array}{|c|}\hline -6 \\ \hline -6 \\ \hline -6 \\ \hline \end{array}$$

the third basic solution is:

$$\left(\begin{array}{|c|}\hline -6 \\ \hline -6 \\ \hline -6 \\ \hline \end{array} \, x + \begin{array}{|c|}\hline -2 \\ \hline 1 \\ \hline 1 \\ \hline \end{array} \right) e^{-x} \equiv \begin{array}{|c|}\hline -6x - 2 \\ \hline -6x + 1 \\ \hline -6x + 1 \\ \hline \end{array} \, e^{-x}$$

3.

$$\mathbb{A} = \begin{array}{|r|r|r|}\hline 42 & -9 & 9 \\ \hline -12 & 39 & -9 \\ \hline -28 & 21 & 9 \\ \hline\end{array}$$

$\Rightarrow \lambda^3 - 90\lambda^2 + 2700\lambda - 27000$ [hint: triple root] $\Rightarrow 6\lambda - 180 = 0$ has a single root of 30 [$\Rightarrow$ triple root of the original polynomial]. Finding eigenvectors:

$$\begin{array}{|r|r|r||}\hline 12 & -9 & 9 \\ \hline -12 & 9 & -9 \\ \hline -28 & 21 & -21 \\ \hline\end{array} \overset{GJ}{\Rightarrow} \begin{array}{|c|c|c||}\hline 1 & -\frac{3}{4} & \frac{3}{4} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \Rightarrow \mathbf{q} = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 0 \\ \hline\end{array} \text{ and } \begin{array}{|r|}\hline -3 \\ \hline 0 \\ \hline 4 \\ \hline\end{array}$$

are the corresponding eigenvectors [only two] which, when multiplied by e^{30x} yield the first two basic solutions.

Squaring the previous matrix yields the zero matrix (any vector is a solution - we just have to make it orthogonal to the previous two). We thus get

$$\begin{array}{|r|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 3 & 4 & 0 \\ -3 & 0 & 4 \\ \hline\end{array} \overset{GJ}{\Rightarrow} \begin{array}{|c|c|c|}\hline 1 & 0 & -\frac{4}{3} \\ \hline 0 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline\end{array} \Rightarrow \mathbf{s} = \begin{array}{|r|}\hline 4 \\ \hline -3 \\ \hline 3 \\ \hline\end{array}$$

which implies

$$\mathbf{q}_s = \begin{array}{|r|r|r||}\hline 12 & -9 & 9 \\ \hline -12 & 9 & -9 \\ \hline -28 & 21 & -21 \\ \hline\end{array} \cdot \begin{array}{|r|}\hline 4 \\ \hline -3 \\ \hline 3 \\ \hline\end{array} = \begin{array}{|r|}\hline 102 \\ \hline -102 \\ \hline -238 \\ \hline\end{array}$$

The third basic solution is thus $\left(\begin{array}{|r|}\hline 102 \\ \hline -102 \\ \hline -238 \\ \hline\end{array}\, x + \begin{array}{|r|}\hline 4 \\ \hline -3 \\ \hline 3 \\ \hline\end{array}\right) e^{30x} = \begin{array}{|c|}\hline 102x + 4 \\ \hline -102x - 3 \\ \hline -238x + 3 \\ \hline\end{array}\, e^{30x}$.

4.

$$\mathbb{A} = \begin{array}{|r|r|r|}\hline -103 & -53 & 41 \\ \hline 160 & 85 & -100 \\ \hline 156 & 131 & -147 \\ \hline\end{array}$$

$\Rightarrow \lambda^3 + 165\lambda^2 + 9075\lambda + 166375$ [hint: triple root] $\Rightarrow 6\lambda + 330 = 0 \Rightarrow \lambda = -55$ [checks] $\Rightarrow$

$$\begin{array}{|r|r|r||}\hline -48 & -53 & 41 \\ \hline 160 & 140 & -100 \\ \hline 156 & 131 & -92 \\ \hline\end{array} \overset{GJ}{\Rightarrow} \begin{array}{|c|c|r||}\hline 1 & 0 & \frac{1}{4} \\ \hline 0 & 1 & -1 \\ \hline 0 & 0 & 0 \\ \hline\end{array} \Rightarrow \mathbf{q} = \begin{array}{|r|}\hline -1 \\ \hline 4 \\ \hline 4 \\ \hline\end{array}$$

is the only eigenvector (this, multiplied by e^{-55x}, provides the first basic solution). Squaring the matrix and adding the extra row, we get

$$\begin{array}{|r|r|r||}\hline 220 & 495 & -440 \\ \hline -880 & -1980 & 1760 \\ \hline -880 & -1980 & 1760 \\ \hline 1 & -4 & -4 \\ \hline \end{array} \overset{GJ}{\Rightarrow} \begin{array}{|r|r|r||}\hline 1 & 0 & -\frac{68}{25} \\ \hline 0 & 1 & \frac{8}{25} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

A solution for $\mathbf{s}$ is thus

$$\begin{array}{|r|}\hline 68 \\ \hline -8 \\ \hline 25 \\ \hline \end{array}$$

which implies that

$$\mathbf{q}_s = \begin{array}{|r|r|r||}\hline -48 & -53 & 41 \\ \hline 160 & 140 & -100 \\ \hline 156 & 131 & -92 \\ \hline \end{array} \cdot \begin{array}{|r|}\hline 68 \\ \hline -8 \\ \hline 25 \\ \hline \end{array} = \begin{array}{|r|}\hline -1815 \\ \hline 7260 \\ \hline 7260 \\ \hline \end{array}$$

The second basic solution is thus

$$(\mathbf{q}_s x + \mathbf{s})e^{-55x} = \begin{array}{|c|}\hline -1815x + 68 \\ \hline 7260x - 8 \\ \hline 7260x + 25 \\ \hline \end{array}\, e^{-55x}$$

Finally, cubing $(\mathbb{A} + 55\mathbb{I})$ results in a zero matrix, so we choose $\mathbf{u}$ to be orthogonal to $[-1, 4, 4]$ and $[68, -8, 25]$, namely

$$\begin{array}{|r|r|r|}\hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline -1 & 4 & 4 \\ \hline 68 & -8 & 25 \\ \hline \end{array} \overset{GJ}{\Rightarrow} \begin{array}{|r|r|r||}\hline 1 & 0 & \frac{1}{2} \\ \hline 0 & 1 & \frac{9}{8} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \Rightarrow \mathbf{u} = \begin{array}{|r|}\hline -4 \\ \hline -9 \\ \hline 8 \\ \hline \end{array}$$

This implies that

$$\mathbf{s}_u = \begin{array}{|r|r|r||}\hline -48 & -53 & 41 \\ \hline 160 & 140 & -100 \\ \hline 156 & 131 & -92 \\ \hline \end{array} \cdot \begin{array}{|r|}\hline -4 \\ \hline -9 \\ \hline 8 \\ \hline \end{array} = \begin{array}{|r|}\hline 997 \\ \hline -2700 \\ \hline -2539 \\ \hline \end{array}$$

and

$$\mathbf{q}_u = \begin{array}{|r|r|r||}\hline -48 & -53 & 41 \\ \hline 160 & 140 & -100 \\ \hline 156 & 131 & -92 \\ \hline \end{array} \cdot \begin{array}{|r|}\hline 997 \\ \hline -2700 \\ \hline -2539 \\ \hline \end{array} = \begin{array}{|r|}\hline -8855 \\ \hline 35420 \\ \hline 35420 \\ \hline \end{array}$$

The last basic solution thus equals

$$\left(\mathbf{q}_u \frac{x^2}{2} + \mathbf{s}_u x + \mathbf{u}\right) e^{-55x} = \begin{array}{|c|}\hline -\frac{8855}{2}x^2 + 997x - 4 \\ \hline 17710x^2 - 2700x - 9 \\ \hline 17710x^2 - 25391x + 8 \\ \hline \end{array}\, e^{-55x}$$