- FIRST-ORDER ODF

Separable:

$$\begin{aligned} y' &= h(x) \cdot g(y) \\ \int \frac{dy}{g(y)} &= \int h(x)dx + C \end{aligned}$$

Scale-independent:

$$\begin{aligned} y' &= g\left(\frac{y}{x}\right) \\ y(x) &= x \cdot u(x) \end{aligned}$$

Linear:

$$\begin{aligned} y' + g(x) \cdot y &= r(x) \\ y(x) &= c(x) \cdot e^{-\int g(x)dx} \end{aligned}$$

Bernoulli:

$$\begin{aligned} y' + f(x) \cdot y &= r(x) \cdot y^a \\ y &= u^{\frac{1}{1-a}} \end{aligned}$$

Exact:

$$\begin{aligned} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy &= 0 \\ f(x,y) &= C \end{aligned}$$

'Nearly' exact:

$$P(x,y)dx + Q(x,y)dy = 0$$

has integrating factor F, given by:

$$\frac{d\ln F}{dx} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}$$

when RHS contains no y, or

$$\frac{d\ln F}{dy} = \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P}$$

when RHS contains no x.

Clairaut:

$$\begin{aligned} y &= xy' + g(y') \\ y &= xC + g(C) \quad \text{regular family of solutions} \end{aligned}$$

$$\left.\begin{array}{c} y = xp + g(p) \\ x = -g'(p) \end{array}\right\} \qquad \text{singular solution}$$

- **Second-order ODE**

  Missing y: Solve for y'.

  Missing x: Solve for y' as a function of y.

  Linear:

  $$y'' + f(x)y' + g(x)y = r(x)$$

  Solve homogeneous version (if you can), getting two basic solutions y_1 and y_2, then get y_p by VP. General solution : $y = Ay_1 + By_2 + y_p$.

  with constant coefficients (meaning f and g): Solve characteristic polynomial, then $y_1 = e^{\lambda_1 x}$, etc. (know how to deal with complex and duplicate roots). Also know how to deal with $r(x) = P(x) \cdot e^{\alpha x}$, where $P(x)$ is a polynomial in x, by technique of *Undetermined Coefficients* (UC).

  Cauchy:

  $$x^2 y'' + a\ x\ y' + b\ y = r(x)$$

  by introducing new *independent* variable $t = \ln x$. Basic solutions to homogeneous version are of the x^m type; know how to set up the corresponding characteristic polynomial for possible values of m.

- **Higher-order linear ODE with constant coefficients**

  Simple extension of the 2^{nd}-order case (only the characteristic polynomial becomes a bit more difficult to solve, but we have Maple). Understand the VP and UC techniques. Also, the Cauchy variant.

- **Set of linear, 1^{st}-order, constant-coefficient equations**

  $$\mathbf{y}' = \mathbb{A}\mathbf{y} + \mathbf{r}(x)$$

  Basic solutions of *homogeneous* version are: $\mathbf{q} \cdot e^{\lambda x}$, where λ and $\mathbf{q}$ are eigenvalues and corresponding eigenvectors of $\mathbb{A}$ (know how to deal with complex and duplicate eigenvalues). When $\mathbf{r}(x) = \mathbf{P}(x) \cdot e^{\alpha x}$, we can construct $\mathbf{y}_p$ by the UC technique (α must not coincide with any λ). Otherwise, we construct $\mathbf{y}_p$ by VP:

$$\mathbf{y}^{(p)} = \mathbb{Y} \int \mathbb{Y}^{-1} \cdot \mathbf{r}(x)\, dx$$

- POWER-SERIES TECHNIQUE

  Be able to solve a *2nd order, linear, homogeneous* differential equation by substituting

  $$y = \sum_{i=0}^{\infty} c_i x^i,$$

  deriving a recurrence formula for the c_i's, solving for the first handful of these, and identifying the pattern if possible. I would usually give you $c_0 = y(0)$ and $c_1 = y'(0)$.

- METHOD OF FROBENIUS

  is used when the coefficients of the previously described equation contain x and/or x^2 in the denominator. Find and solve the indicial equation for r, substitute

  $$y = \sum_{i=0}^{\infty} c_i x^{i+r}$$

  and follow the same steps as in the previous case. This time, I would usually ask only for the first basic solution (with the bigger r).

- Be able to (in the same type of equation) introduce a new *dependent* variable, followed by introducing a new *independent* variable. The equation should then turn into an equation (usually Bessel) for which the solution is known.

- A point, vector, straight line and plane in 3D, and how are they represented in Cartesian (usual) coordinates. Dot and cross product (of two vectors), and how they can help to establish the (shortest) distance between two 'objects' (points, straight lines, planes - take all combinations). For the line-plane and plane-plane combination, the shortest distance is 0 unless they are parallel (can you tell whether they are?). Areas of triangles and volumes of tetrahedrons. Vector can also be multiplied by a scalar (elementwise).

- Curves

  Be able to parametrize at least a *straight line* (in 3D) and a *circle* (in one of the coordinate planes). For a given curve (in its parametric form - seen as a motion of a particle), find velocity, speed, acceleration (tangential and normal). Also (at a given point) curvature and torsion.

- Gradient of a scalar field, and divergence and curl of a vector field. Find a *normal* vector to a $f(x, y, z) = c$ surface. Directional derivative of a scalar field.

- Type I Line Integrals

  One must first parametrize the curve, than do:

  $$\int\limits_C f(x, y, z)\ ds = \int_{t_1}^{t_2} f(\mathbf{r}(t))|\dot{\mathbf{r}}(t)|\ dt$$

  (independent of parametrization).

  Curve's length

  $$L = \int\limits_C ds$$

  center of mass

  $$\left[\frac{\int\limits_C x\ ds}{L}, \frac{\int\limits_C y\ ds}{L}, \frac{\int\limits_C z\ ds}{L}\right]$$

  and moment of inertia

  $$\frac{M}{L}\int\limits_C d(x, y, z)^2\ ds$$

  where $d(x, y, z)$ is the distance from a given axis (shortest distance from $[x, y, z]$ to a straight line - remember?) and M is the curve's mass (when they tell you that the curve has a unit mass density, take $\frac{M}{L} = 1$).

  Average value of $f(x, y, z)$ over a curve

  $$\frac{\int\limits_C f(x, y, z)\ ds}{L}$$

- Type II line integrals

  One must first parametrize the curve, then do:

$$\int_C \mathbf{g}(x,y,z) \bullet d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{g}(\mathbf{r}(t)) \bullet \dot{\mathbf{r}}(t)\ dt$$

  (also independent of parametrization).

  Path independent when $curl(\mathbf{g}) = \mathbf{0}$, in which case

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{g}(x,y,z) \bullet d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$$

  where **a** and **b** are two given points, and $f(x,y,z)$ is such that its gradient equals **g** (be able to find f). Over a *closed* curve, this integral would have to equal to 0 (right?).

- Double Integrals

$$\iint_{\mathcal{R}} f(x,y)\ dxdy$$

  ($dxdy$ is sometimes, quite appropriately, replaced by dA, denoting an infinitesimal area) are evaluated by converting them into two consecutive univariate (i.e. having a single variable, the usual) integrals, the first (inner) with respect to x (with conditional limits which may depend on y), the second with respect to y (with marginal, 'shadow' limits). Or the other way round (the two results must be the same).

  Area of $\mathcal{R}$

$$A = \iint_{\mathcal{R}} dxdy$$

  its center of mass

$$\left[\frac{\iint_{\mathcal{R}} x\ dxdy}{A}, \frac{\iint_{\mathcal{R}} y\ dxdy}{A} \right]$$

  and moment of inertia

$$\frac{M}{A} \iint_{\mathcal{R}} d(x,y)^2 dxdy$$

where M is the total mass, and d is the distance from axis of rotation. Unit mass density means $\frac{M}{A} = 1$.

Volume

$$\iint_{\mathcal{R}} h(x, y)\ dxdy$$

where $h(x, y)$ is the (3D) object's 'thickness' (in the z direction), and $\mathcal{R}$ is its 'shadow' in the x-y plane.

- TYPE I SURFACE INTEGRALS

  One must first parametrize the surface, then do:

  $$\iint_{S} f(x, y, z)\ dA = \iint_{\mathcal{R}} f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\ dudv$$

  (independent of parametrization).

  Surface's area

  $$A = \iint_{S} dA$$

  center of mass

  $$\left[\frac{\iint_S x\ dA}{A}, \frac{\iint_S y\ dA}{A}, \frac{\iint_S z\ dA}{A}\right]$$

  and moment of inertia

  $$\frac{M}{A} \iint_{S} d(x, y, z)^2\ dA$$

  where $d(x, y, z)$ is the distance from a given axis, and A is the surface's mass (when they tell you that the surface has a unit mass density, take $\frac{M}{L} = 1$).

  Average value of $f(x, y, z)$ over a surface

  $$\bar{f} \equiv \frac{\iint_S f(x, y, z)\ dA}{A}$$

- TYPE II SURFACE INTEGRALS

One must first parametrize the surface, then do:

$$\iint\limits_{S} \mathbf{g}(x,y,z) \bullet d\mathbf{A} = \iint\limits_{\mathcal{R}} \mathbf{g}(\mathbf{r}(u,v)) \bullet (\mathbf{r}_u \times \mathbf{r}_v)\ du dv$$

(also independent of parametrization).

- Triple (volume) integrals

  are only of one (scalar) type, namely

$$\iiint\limits_{\mathcal{V}} f(x,y,z) dV$$

  where $\mathcal{V}$ is a 3D region. They can be converted into three consecutive univariate integrals. To start with, say, the dz integration, one has to solve the equation(s) for the surface (sides) of $\mathcal{V}$ for z, to get the integral's limits. One then has to project $\mathcal{V}$ into the x-y plane, and integrate the result of the dz integration over this projection (a usual 2D integral).

  The integration is sometimes simplified by introducing (three) new variables, say θ, ϕ and r - the so called spherical coordinates being most common. Don't forget that $dxdydz$ must be replaced by the Jacobian, multiplied by $d\theta d\phi dr$ (for spherical coordinates, the Jacobian is $r^2 \sin\theta$).

  Using triple integrals, one can find a *volume, center of mass, moment of inertia* of any 3D region. Also, an *everage* value of $f(x,y,z)$ over such region.

- Gauss (divergence) Theorem

  There is an interesting connection between a Type-II surface integral over a *closed* surface, and a volume integral of the corresponding divergence (over the 3D region enclosed by the surface):

$$\iint\limits_{\mathcal{S}_c} \mathbf{g} \bullet\ d\mathbf{A} \equiv \iiint\limits_{\mathcal{V}} \mathrm{Div}(\mathbf{g})\, dV$$

- Stokes Theorem

Similarly, there is a connection between a Type-II line integral over a *closed* curve $\mathcal{C}_d$, and a (Type-II) surface integral of the corresponding curl (over any surface with $\mathcal{C}_d$ as boundaries):

$$\oint_{\mathcal{C}_d} \mathbf{g} \bullet \; d\mathbf{r} \equiv \iint_{\mathcal{S}} \mathrm{Curl}(\mathbf{g}) \bullet \mathbf{n} \, dA$$

- Analytic functions - Differentiation

  Understand the four basic operations (+, -, *, /) on complex numbers, also polar representation of a complex number. Also the fact that complex numbers can be seen as vectors in 2D.

  Be able to *evaluate* a complex expression (or a function, at a specific z), e^z, $\mathrm{Ln}(z)$, and $\sqrt[n]{z}$ in particular.

  Be able to tell when a function of z is analytic (easy - it must not contain conjugation, Re, Im, or $|..|$), and where are its singularities and branch points. Its derivative is then the same as if z were real. An analytic function of $z = x + iy$ can be written as $u(x, y) + i\; v(x, y)$, where both u and v are *harmonic* (be able to find v given u, or u given v, based on Rieman-Cauchy equations).

- Complex Integration

  To evaluate

$$\int_C f(z) \; dz$$

  we have to parametrize C, i.e. find $z(t)$ which takes us through C when t varies from t_1 to t_2, and then compute

$$\int_{t_1}^{t_2} f(z(t)) \; \dot{z}(t) \; dt$$

  (thus evaluating two *ordinary* integrals, as the integrand consists of Re and Im parts - both real functions of t).

  This is necessary only when $f(z)$ is *not* analytic. For fully analytic (entire) functions, we can do instead

$$\int_C f(z) \; dz = F(z)|_{z=z_1}^{z_2}$$

where $F(z)$ is the usual antiderivative (indefinite integral) of $f(z)$, found as if z were real, and z_1 and z_2 are the initial and final points of C. This type of integral is thus path independent.

- CONTOUR INTEGRATION

  When $f(z)$ is an analytic function with a few SINGULARITIES, we can evaluate

  $$\oint_C f(z)\, dz$$

  over any *closed* path (traversed counterclockwise) by finding the residue of each singularity inside C, and multiplying their *sum* by $2\pi i$.

  The residue of $f(z)$ at z_0 is found by

  $$\lim_{z \to z_0} (z - z_0) f(z)$$

  (with the help of L'Hopital rule, if necessary), when dealing with a first-order singularity (most common), and

  $$\frac{\lim_{z \to z_0} [(z - z_0)^m f(z)]^{(m-1)}}{(m-1)!}$$

  for an m^{th}-order singularity. The value of m can be established by starting the above computation with $m = 1$, and moving to a higher m each time we get an infinite answer.

  This can help evaluating *real* integrals of the following type

  $$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$$

  where P and Q are two polynomials (the degree of Q must be higher than that of P by at least 2, and Q must have no *real* roots), and

  $$\int_0^{2\pi} R(\sin t, \cos t) dt$$

  where R is a rational function (whose denominator must have no real roots).