Set 4.4
Question 4:
Indicial equation yields $r_1 = 0$ and $r_2 = \frac{1}{2}$ (Case I).
Substituting the usual trial solution yields:

$$2\sum_{i=0}^{\infty}(r+i)(2r+2i-1)c_i x^{r+i-1} + \sum_{i=1}^{\infty} c_{i-1}x^{r+i-1}$$

implying

$$\begin{array}{cc}
r = 0 & r = \frac{1}{2} \\
c_i = \dfrac{-c_{i-1}}{2i(2i-1)} & c_i = \dfrac{-c_{i-1}}{(2i+1)2i} \\
c_0 = 1 & c_0 = 1 \\
c_1 = -\frac{1}{2} & c_1 = -\frac{1}{3!} \\
c_2 = \frac{1}{4!} & c_2 = \frac{1}{5!} \\
c_3 = -\frac{1}{6!} & c_3 = -\frac{1}{7!} \\
c_4 = \frac{1}{8!} & c_4 = \frac{1}{9!} \\
\ldots & \ldots
\end{array}$$

The two basic solutions are:

$$1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \ldots = \cos\sqrt{x}$$

and

$$\left(x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \frac{x^{9/2}}{9!} - \ldots\right) = \sin\sqrt{x}$$

Question 6:
Indicial equation yields $r_1 = 0$ and $r_2 = -1$ (Case III).
Substituting the usual trial solution yields:

$$\sum_{i=0}^{\infty}(r+i)(r+i+1)c_i x^{r+i-1} + 4\sum_{i=2}^{\infty} c_{i-2}x^{r+i-2}$$

implying, for $r = 0$

$$c_i = \frac{-4c_{i-2}}{i(i+1)}$$
$$c_0 = 1$$
$$c_2 = -\tfrac{4}{3!}$$
$$c_4 = \tfrac{4^2}{5!}$$
$$c_6 = -\tfrac{4^3}{7!}$$
$$c_8 = \tfrac{4^4}{9!}$$
$$...$$

$(c_1 = c_3 = c_5 = ... = 0)$. The first basic solution is thus $y_1 = \sum_{k=0}^{\infty} \frac{(-4)^k x^{2k}}{(2k+1)!} = \dfrac{\sin(2x)}{2x}$.

Substituting

$$\sum_{i=0}^{\infty} c_i x^{i-1} + k y_1 \ln x$$

into the equation yields $\frac{k}{x} + ...$ The only way to make the x^{-1} term equal to zero is to take $k = 0$. After that, it is quite simple to do

$$c_i = \frac{-4c_{i-2}}{(i-1)i}$$
$$c_0 = 1$$
$$c_2 = -\tfrac{4}{2!}$$
$$c_4 = \tfrac{4^2}{4!}$$
$$c_6 = -\tfrac{4^3}{6!}$$
$$c_8 = \tfrac{4^4}{8!}$$
$$...$$

which implies that the second basic solution is $\sum_{k=0}^{\infty} \frac{(-4)^k x^{2k-1}}{(2k)!} = \dfrac{\cos(2x)}{x}$.

Question 8. Indicial equation yields $r = -1$ and 0 (Case III).

For $r = 0$ we get $c_1 = c_0 = 1$, and

$$c_i = \frac{2ic_{i-1} - c_{i-2}}{i(i+1)}$$
$$c_2 = \frac{1}{2}$$
$$c_3 = \frac{1}{3!}$$
$$c_4 = \frac{1}{4!}$$
$$c_5 = \frac{1}{5!}$$
$$c_6 = \frac{1}{6!}$$
$$...$$

resulting in $y_1 = e^x$. For the same reason as in the previous example, we have to take $k = 0$. And, for the second c sequence, we get: $c_0 = c_1 = 1$ (c_1 can have any value)

$$c_i = \frac{2(i-1)c_{i-1} - c_{i-2}}{i(i-1)}$$
$$c_2 = \frac{1}{2}$$
$$c_3 = \frac{1}{3!}$$
$$c_4 = \frac{1}{4!}$$
$$c_5 = \frac{1}{5!}$$
$$c_6 = \frac{1}{6!}$$
$$...$$

implying $\dfrac{e^x}{x}$ for the second basic solution.

Question 12: Indicial equation yields $r = 0$ (double - Case II). Substituting yields $c_1 = c_3 = c_5 = ... = 0$ and

$$c_i = \frac{c_{i-2}}{i^2}$$
$$c_2 = \frac{1}{2^2}$$
$$c_4 = \frac{1}{2^4(2!)^2}$$
$$c_6 = \frac{1}{2^6(3!)^2}$$
$$c_8 = \frac{1}{2^8(4!)^2}$$
$$c_{10} = \frac{1}{2^{10}(5!)^2}$$
$$...$$

which Maple helps to identify as the following Bessel function: $\sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2} = I_0(x)$.

Chapter 4 Review:

Question 22: Indicial equation yields $r = 0$ (double), matching the coefficients gives $c_1 = c_0 = 1$ and

$$c_i = \frac{(2i-1)c_{i-1} - c_{i-2}}{i^2}$$

$$c_2 = \frac{1}{2!}$$

$$c_3 = \frac{1}{3!}$$

$$c_4 = \frac{1}{4!}$$

$$c_5 = \frac{1}{5!}$$

$$\cdots$$

implying that $y_1 = e^x$. Substituting $y_1 \ln x$ into the equation results in 0, $e^x \ln x$ is therefore the second basic solution.