1.

$$\begin{aligned}
x^3 + 2ye^x &= xe^x y' \\
xy' - 2y &= x^3 e^{-x} \\
\frac{dy}{y} &= 2\frac{dx}{x} \\
y &= c(x) \cdot x^2 \\
x^3 c' &= x^3 e^{-x} \\
c(x) &= C - e^{-x} \\
y &= Cx^2 - x^2 e^{-x}
\end{aligned}$$

Answer:

$$x = y^2(C - e^{-y})$$

2.

$$m(m-1) - 3m + 4 = m^2 - 4m + 4$$

yields $m = 2$ as a double root. Substituting

$$y_p = (A \ln x + B)x$$

in the original equation yields (after dividing by x):

$$(A-1)\ln x + B - 2A = 0$$

which implies $A = 1$ and $B = 2$. Solution:

$$y = x^2(C_1 + C_2 \ln x) + x(2 + \ln x)$$

3. Bernoulli, $a = 2$. Introducing $u = \frac{1}{y}$, we get

$$\begin{aligned}
u' - \frac{2x-1}{x^2}u &= 1 \\
\frac{du}{u} &= \left(\frac{2}{x} - \frac{1}{x^2}\right) dx \\
\ln u &= 2\ln x + \frac{1}{x} + \ln c \\
u &= c(x) \cdot x^2 e^{1/x} \\
c' &= x^{-2} e^{-1/x} \\
c(x) &= e^{-1/x} + C \\
u &= x^2(Ce^{1/x} + 1) \\
y &= \frac{1}{x^2(1 + Ce^{1/x})}
\end{aligned}$$

4. $z = y'$

$$z' = \frac{z}{x}(1 + \ln\frac{z}{x})$$

$u = \frac{z}{x}$

$$\begin{aligned} u'x &= u \ln u \\ \frac{du}{u \ln u} &= \frac{dx}{x} \\ \ln\ln u &= \ln x + \ln C_1 \\ \ln u &= C_1 \cdot x \\ u &= e^{C_1 x} \\ z &= x e^{C_1 x} \\ y &= \frac{C_1 x - 1}{C_1^2} \cdot e^{C_1 x} + C_2 \end{aligned}$$

5. $P = y \cos\frac{x}{y} + 2xy^2$, $Q = -x\cos\frac{x}{y} - y^2$.

$$\begin{aligned} \frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} &= -\frac{2}{y} \\ F &= e^{-2\ln y} = \frac{1}{y^2} \\ G &= \int \left(\frac{1}{y}\cos\frac{x}{y} + 2x\right) dx = \sin\frac{x}{y} + x^2 \\ f &= G + \int \left(-\frac{x}{y^2}\cos\frac{x}{y} - 1 - \frac{\partial G}{\partial y}\right) dy = \sin\frac{x}{y} + x^2 - y \end{aligned}$$

Solution

$$\sin\frac{x}{y} + x^2 - y = C$$