Set 2.14

12: Characteristic polynomial has clearly $\lambda = 1$ as one root, which leads to

$$(\lambda^3 - 3\lambda^2 + 3\lambda - 1) \div (\lambda - 1) = \lambda^2 - 2\lambda + 1$$

telling us that all 3 roots are equal to 1. Thus

$$y := (C_1 + C_2 x + C_3 x^2) e^x$$

Since

$$\begin{aligned} y(0) &= C_1 = 2 \\ y'(0) &= C_1 + C_2 = 2 \\ y''(0) &= C_1 + 2C_2 + 2C_3 = 10 \end{aligned}$$

we get (by solving these):

$$y := (2 + 4x^2) e^x$$

Set 2.15

2:

$$m(m-1)(m-2) + m(m-1) - 2m + 2 = m^3 - 2m^2 - m + 2$$

Again, $m_1 = 1$ is one root, leading to

$$m^2 - m - 2$$

and $m_2 = -1$, $m_3 = 2$.

The particular solution has the form of (by-passing t, which is OK):

$$y_p = (A \ln x + B) \cdot x^3$$

This, substituted into the original equation yields:

$$(8A - 1) \ln x + 8B + 14A = 0$$

Solution:

$$y = C_1 x + C_2 x^2 + \frac{C_3}{x} + \left(\frac{\ln x}{8} - \frac{7}{32} \right) x^3$$

4: Again, $\lambda_1 = 1$ is clearly a solution, which leads to

$$\lambda^2 + 3\lambda + 2$$

and $\lambda_2 = -1$, $\lambda_3 = -2$. The particular solution is now $y_p = Ax^3 + Bx^2 + Cx + D$. Substituted, we get:

$$(4 - 2A)x^3 - (3A + 2B)x^2 + (12A - 2B - 2C)x + 6A + 4B - C - 2D - 1 = 0$$

which can be solved easily for $A = 2$, $B = -3$, $C = 15$, $D = -8$. Solution:

$$y := C_1 e^x + C_2 e^{-x} + C_3 e^{-2x} + 2x^3 - 3x^2 + 15x - 8$$

6: Trying for multiple root, we get

$$\gcd(\lambda^3 - 6\lambda^2 + 12\lambda - 8, 3\lambda^2 - 12\lambda + 12) = \lambda^2 - 4\lambda - 4$$

All three roots are therefore equal to 2. To find u', v' and w', we have to solve (dividing each equation by e^{2x}):

$$\begin{bmatrix} 1 & x & x^2 \\ 2 & 1+2x & 2x+2x^2 \\ 4 & 4+4x & 2+8x+4x^2 \end{bmatrix} \cdot \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{x} \end{bmatrix}$$

Gaussian elimination converts this to:

$$\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{x} \end{bmatrix}$$

which yields (backward substitution): $w' = \dfrac{\sqrt{x}}{2}$, $v' = -x^{3/2}$, $u' = \dfrac{x^{5/2}}{2}$. Integrating and plugging into the $u \cdot y_1 + v \cdot y_2 + w \cdot y_3$ formula yields:

$$y_p = \frac{8x^{7/2}e^{2x}}{105}$$

General solution:

$$C_1 e^{2x} + C_2 x e^{2x} + C_3 x^2 e^{2x} + \frac{8x^{7/2}e^{2x}}{105}$$

Chapter 2 review

25: First

$$(\lambda^2)_{1,2} = \frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{16}{4}} = 4,\ 1$$

which means the four roots are ± 1 and ± 2. Particular solution has the form of

$$A \sin 2x + B \cos 2x$$

Substituted into the original equation, we get:

$$40A \sin 2x + (40B - 40)\cos 2x = 0$$

which implies that $A = 0$ and $B = 1$. Solution:

$$C_1 e^x + C_2 e^{-x} + C_3 x^{2x} + C_4 e^{-2x} + \cos 2x$$