1. Scale independent
$$y' = \frac{y}{x} + \frac{y}{x}\ln(\frac{y}{x})$$
Substitution $y = x \cdot u$ leads to
$$xu' = u \ln u$$
(separable)
$$\begin{aligned} \frac{du}{u \ln u} &= \frac{dx}{x} \\ \ln \ln u &= \ln x + \tilde{c} \\ \ln u &= cx \\ u &= e^{cx} \\ y &= xe^{cx} \end{aligned}$$

2. Clairaut
$$y = xy' - y' \ln(y')$$
Regular solutions
$$y = Cx - C \ln C$$
Since $g(p) = -p \ln p$, we get
$$\begin{aligned} x - -g'(p) &= \ln p + 1 \\ p &= e^{x-1} \\ y &= xe^{x-1} - e^{x-1}(x-1) = e^{x-1} \end{aligned}$$
(singular).

3. Missing y (or Cauchy, but that makes it more difficult). Introducing $z = y'$
$$xz' = z + \ln x$$
(linear). Solving homogeneous
$$\begin{aligned} xz' &= z \\ \frac{dz}{z} &= \frac{dx}{x} \\ \ln z &= \ln x + \tilde{c} \\ z &= C \cdot x \end{aligned}$$
Substitute into full equation
$$\begin{aligned} x(C'x + C) &= Cx + \ln x \\ C' &= \frac{\ln x}{x^2} \\ C &= -\frac{\ln x}{x} - \frac{1}{x} + c_1 \end{aligned}$$

which implies

$$\begin{aligned} z &= y' = -\ln x - 1 + c_1 x \\ y &= -x\ln x + c_1\frac{x^2}{2} + c_2 = -x\ln x + \tilde{c}_1 x^2 + c_2 \end{aligned}$$

4. Cauchy

$$x^2y'' - xy' + y = x\ln x$$

Characteristic polynomial is

$$m(m-1) - m + 1 = m^2 - 2m + 1 = (m-1)^2$$

The two basic solutions (to homogeneous part) are x and $x\ln x$. Using undetermined coefficients:

$$y_p = (A\ln x + B)x \cdot (\ln x)^2$$

Substituting, one gets

$$(6A - 1)\ln x + 2B = 0$$

which implies that $A = \frac{1}{6}$ and $B = 0$. Answer:

$$y = c_1 x + c_2 x \ln x + \frac{x}{6}(\ln x)^3$$

5. Characteristic polynomial is

$$\lambda^4 + 4\lambda^3 - 2\lambda^2 - 12\lambda + 9$$

divided by $\lambda - 1$:

$$\lambda^3 + 5\lambda^2 + 3\lambda - 9$$

further divided by $\lambda - 1$:

$$\lambda^2 + 6\lambda + 9$$

so the four roots are 1, 1, 3 and 3. The homogeneous solution is

$$y_h = c_1 e^x + c_2 x e^x + c_3 e^{-3x} + c_4 x e^{-3x}$$

Using undetermined coefficients

$$y_p = (Ax + B)e^x \cdot x^2$$

Substituting (with the help of Maple), we get

$$(96A - 1)x + 48A + 32B = 0$$

implying that $A = \frac{1}{96}$ and $B = -\frac{48}{32}A = -\frac{1}{64}$. Solution:

$$y = c_1 e^x + c_2 x e^x + c_3 e^{-3x} + c_4 x e^{-3x} + \left(\frac{x^3}{96} - \frac{x^2}{64}\right)e^x$$

6. Cauchy. Characteristic polynomial is

$$\begin{aligned}&m(m-1)(m-2)(m-3)+11m(m-1)(m-2)+28m(m-1)+8m-8\\&=m^4+5m^3+6m^2-4m-8\end{aligned}$$

Dividing by $m-1$:

$$m^3+6m^2+12m+8$$

The greatest common divisor of this polynomial and its derivative is

$$(m+2)^2$$

implying that 1 is a single root and -2 is a *triple* root. Thus

$$y=c_1x+x^{-2}[c_2+c_3\ln x+c_4(\ln x)^2]$$

To meet the initial conditions, we need (use Maple to evaluate derivatives)

$$\begin{aligned}c_1+c_2&=3\\c_1-2c_2+c_3&=2\\6c_2-5c_3+2c_4&=0\\-24c_2+26c_3-18c_3&=-1\end{aligned}$$

which can be solved with the help of 'linsolve', to yield $c_1=\frac{61}{27}$, $c_2=\frac{20}{27}$, $c_3=\frac{11}{9}$ and $c_4=\frac{5}{6}$. Thus

$$y=\frac{61x}{27}+x^{-2}\left[\frac{20}{27}+\frac{11}{9}\ln x+\frac{5}{6}(\ln x)^2\right]$$