1. Find the L-U decomposition of

$$\mathbb{A} \equiv \begin{bmatrix} 2 & -1 & 0 \\ 3 & -2 & 5 \\ 0 & -3 & 4 \end{bmatrix}$$

Use the results to find the three-component vector $\mathbf{x}$ which solves

$$\mathbb{A}\,\mathbf{x} = \begin{bmatrix} 7 \\ 17 \\ 13 \end{bmatrix}$$

2. Using

$$y = a \cdot x + b \cdot 2^x$$

as your model, find the best (least-square) values of a and b based on the following data

x:	-2	0	3	7	13
y:	6	0	-9	-20	43

3. Find a, b and c to minimize

$$\sum_{i=1}^{5} \left[y_i - (a + b \cdot x^2 + c \cdot x^4)\right]^2$$

where

x	-2	-1	0	2	3
y	2.6	3.8	4.5	5.0	3.9

4. Using Newton's technique, find the interpolating polynomial for

x	1.1	1.2	1.3	1.4
y	0.47145230	0.43672247	0.39736138	0.35505991

5. Fit the best (least square) quadratic polynomial to

x	10	12	14	16	18	20
y	112	96	83	91	103	99

6. Fit a cubic spline to the following set of points:

year:	1991	1995	1997	2001
inflation:	1.36	1.58	1.20	1.41

Solution (not yet verified):

1.

$$
\begin{aligned}
\mathbb{A} &= \begin{bmatrix} 2 & 0 & 0 \\ 3 & -\dfrac{1}{2} & 0 \\ 0 & -3 & -26 \end{bmatrix} \begin{bmatrix} 1 & -\dfrac{1}{2} & 0 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{x} &= \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}
\end{aligned}
$$

2.

$$y = -3.03629311\ x + 0.100672967 \times 2^x$$

3.

$$y = 4.299418594 - 0.218095923\ x^2 + 0.01954941788\ x^4$$

4.

$$
\begin{aligned}
y &= 0.4714523 - 0.3472983(x-1.1) + 0.231563(x-1.1)(x-1.2) \\
&\quad +0.28181\overline{3}(x-1.1)(x-1.2)(x-1.3)
\end{aligned}
$$

5.

$$y = 89 - \frac{18}{35}(x-15) + \frac{5}{7}(x-15)^2$$

6.

$$
\begin{array}{rcl}
\dfrac{34}{25} + \dfrac{107}{700}(x-1991) - \dfrac{137}{22400}(x-1991)^3 & 1991 < & x < 1995 \\
\dfrac{79}{50} - \dfrac{197}{1400}(x-1995) - \dfrac{411}{5600}(x-1995)^2 + \dfrac{39}{1600}(x-1995)^3 & 1995 < & x < 1997 \\
\dfrac{6}{5} - \dfrac{397}{2800}(x-1997) + \dfrac{51}{700}(x-1997)^2 - \dfrac{17}{2800}(x-1997)^3 & 1997 < & x < 2001
\end{array}
$$

Second midterm:

1. Utilizing the symmetry, construct the first five (up to and including ϕ_4) polynomials, orthogonal in the following sense:

$$\int_{-1}^{1} \sqrt{1-x^2}\phi_i(x)\phi_j(x)dx = 0 \qquad \text{whenever} \quad i \neq j$$

2. Use the results of the previous question to find a cubic polynomial $p(x)$ which minimizes

$$\int_{-1}^{1} \sqrt{1-x^2} \cdot [p(x) - \sin x]^2 \, dx$$

3. Use the results of Question 1 to construct the 4 point Gausssian formula to approximate

$$\int_{-1}^{1} \sqrt{1-x^2} \cdot y(x)dx$$

   Apply your formula to

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+e^x} dx$$

4. Construct a formula to approximate $y'(x_0)$ based on $y(x_0-h)$, $y(x_0)$, $y(x_0+h)$ and $y(x_0+3h)$.

5. Find a cubic polynomial, say $p(x)$, which minimizes

$$\int_{-1}^{1} [p(x) - |x|]^2 \, dx$$

6. Using the composite trapezoidal rule with $n = 2$, 4 and 8, followed by all stages of Romberg's algorithm, approximate

$$\int_{1}^{7} \frac{\exp(\sqrt{x})}{x^2+1} dx$$

7. Develop a formula for approximating

$$\int_0^4 y(x)\,dx$$

based on $y(1)$, $y(2)$ and $y(3)$. Apply the formula to

$$\int_0^4 \cos\left(\frac{x^2}{10}\right) dx$$

Solution:

1. $\phi_0(x) = 1 \qquad \alpha_0 = \frac{\pi}{2}$

   $\phi_1(x) = x \qquad \alpha_1 = \frac{\pi}{8}$

   $\phi_2(x) = x^2 - \frac{1}{4} \qquad \alpha_2 = \frac{\pi}{32}$

   $\phi_3(x) = x^3 - \frac{x}{2} \qquad \alpha_3 = \frac{\pi}{128}$

   $\phi_4(x) = x^4 - \frac{3}{4}x^2 + \frac{1}{16} \qquad \alpha_4 = \frac{\pi}{512}$

2. $a = 0$, $b = \frac{8}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\cdot x \cdot \sin x \; dx = 0.9192278796$, $c = 0$, $d = \frac{128}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\cdot(x^3 - \frac{x}{2})\cdot \sin x \; dx = -0.1585048937 \Rightarrow p(x) = 0.9192278796x - 0.1585048937(x^3 - \frac{x}{2}) = 0.9984803265x - 0.1585048937x^3$

3. Solving $\phi_4(x) = 0$ yields: $x_1 = \frac{-1-\sqrt{5}}{4}$, $x_2 = \frac{1-\sqrt{5}}{4}$, $x_3 = \frac{-1+\sqrt{5}}{4}$ and $x_4 = \frac{1+\sqrt{5}}{4}$. Solving $\begin{array}{c} 2c_1 + 2c_2 = \int_{-1}^{1}\sqrt{1-x^2}dx = \frac{\pi}{2} \\ 2c_1x_1^2 + 2c_2x_2^2 = \int_{-1}^{1}x^2\sqrt{1-x^2}dx = \frac{\pi}{8} \end{array}$ yields $c_1 = \frac{5-\sqrt{5}}{40}\pi$ and $c_2 = \frac{5+\sqrt{5}}{40}\pi \Rightarrow \int_{-1}^{1}\sqrt{1-x^2}\cdot y(x)dx \simeq \frac{5-\sqrt{5}}{40}\pi \cdot \left[y\left(\frac{-1-\sqrt{5}}{4}\right) + y\left(\frac{1+\sqrt{5}}{4}\right)\right] + \frac{5+\sqrt{5}}{40}\pi\left[y\left(\frac{1-\sqrt{5}}{4}\right) + y\left(\frac{-1+\sqrt{5}}{4}\right)\right]$. When $y(x) = \frac{1}{1+e^x}$, this results in 0.7853981634 (an amazingly accurate answer).

4. Solving $\begin{array}{l} c_{-1}+c_0+c_1+c_3=0 \\ -c_{-1}+c_1+3c_3=1 \\ c_{-1}+c_1+9c_3=0 \\ -c_{-1}+c_1+27c_3=0 \end{array}$, we get $c_{-1}=-\dfrac{3}{8}$, $c_0=-\dfrac{1}{3}$, $c_1=\dfrac{3}{4}$ and $c_3=-\dfrac{1}{24} \Rightarrow y'(x_0) \simeq \dfrac{-9y(x_0-h)-8y(x_0)+18y(x_0+h)-y(x_0+3h)}{24h}$

5. $a=\dfrac{1}{2}\int_{-1}^{1}|x|\ dx=\dfrac{1}{2}$, $b=0$, $c=\dfrac{45}{8}\int_{-1}^{1}|x|\,(x^2-\dfrac{1}{3})\ dx=\dfrac{15}{16}$ and $d=0 \Rightarrow p(x)=\dfrac{1}{2}+\dfrac{15}{16}(x^2-\dfrac{1}{3})=\dfrac{3}{16}+\dfrac{15}{16}x^2$

6. 3.76548335 3.26385365 3.257452448
3.389261076 3.257852522
3.290704661

7. $\int_0^4 y(x)dx \simeq \dfrac{8}{3}y(1)-\dfrac{4}{3}y(2)+\dfrac{8}{3}y(3)$, $\int_0^4 \cos\left(\dfrac{x^2}{10}\right)dx \simeq 3.082889698$

Last year final exam (selected questions):

1. Fit a natural cubic spline to the following data:

x	–3	0	2	6
y	4	–1	3	2

Show all details of solving the linear equations for c_2 and c_3 (do not use Maple).

2. Find the $\mathbb{LU}$ decomposition of

$$\mathbb{A} \equiv \begin{bmatrix} 3 & -2 & 0 & 0 \\ 1 & 0 & -5 & 0 \\ 0 & 4 & 0 & -1 \\ 0 & 0 & 3 & -6 \end{bmatrix}$$

Use the result to find a solution to

$$\mathbb{A}\,\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 5 \end{bmatrix}$$

3. Construct a formula to approximate $y'(x_0)$ based on $y(x_0 - h)$, $y(x_0)$, $y(x_0 + h)$ and $y(x_0 + 3h)$. Note that we are skipping $y(x_0 + 2h)$; also note that the resulting formula must yield the exact answer for *all* cubic polynomials.

   Use your formula to find $y'(1.14)$, given that

x	$y(x)$
1.13	0.80036 477
1.14	0.79704 693
1.15	0.79370 777
...	...
1.17	0.78696 632

4. Use Simpson's rule with $n = 4$, 8, 16 and 32, and all (in this case three) stages of the Romberg algorithm to evaluate

$$\int_0^3 \frac{\sin(x)}{x}\,dx$$

5. Solve the following equations

$$\begin{aligned}\frac{x_1 + x_2 + x_3}{3} &= 7\\ \sqrt[3]{x_1 x_2 x_3} &= 4\\ \frac{3}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}} &= \frac{16}{7}\end{aligned}$$

   for x_1, x_2 and x_3 (note that 7, 4 and $\frac{16}{7}$ represent the algebraic, geometric and harmonic mean, respectively, of the three numbers). Apply five iterations of Newton's technique, starting with $x_1 = 2$, $x_2 = 4$ and $x_3 = 10$ as initial values.

6. Solve, approximately

$$\exp(\frac{x}{2}) \cdot y'' = y' - x \cdot y - \ln(x)$$

   where $y(1) = 3$ and $y(4) = -1$, by dividing the $[1, 4]$ interval into three, and then six, subintervals. Improve the values of $y(2)$ and $y(3)$ by Richardson extrapolation.

7. Solve, approximately

$$y'' = \frac{y \cdot \sin(y')}{2-x} + \exp(x^2)$$

where $y(0) = 1$ and $y(1) = 2$, by subdividing the $[0, 1]$ interval into five subintervals.

8. Using Newton's technique, find a solution to

$$\begin{aligned} \frac{x^2}{1+y} - y^3 &= 24 \\ y(1+x^2) - 3xy &= \frac{5}{2} \end{aligned}$$

9. Find a tridiagonal matrix with the same eigenvalues as

$$\begin{bmatrix} 3 & 1 & 4 & 2 \\ 1 & -2 & 3 & -1 \\ 4 & 3 & 4 & -2 \\ 2 & -1 & -2 & 1 \end{bmatrix}$$

10. Perform three iterations of the QR algorithm with shift, using the following matrix:

$$\begin{bmatrix} 3 & -2 & 0 \\ -2 & 1 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

Based on the results, make the best possible estimate of the corresponding three eigenvalues.

Solution:

1.

$$\begin{array}{rcl} 4 - \frac{2033}{696}(x+3) + \frac{97}{696}(x+3)^3 & -3 < & x < 0 \\ -1 + \frac{293}{348}x + \frac{291}{232}x^2 - \frac{235}{696}x^3 & 0 < & x < 2 \\ 3 + \frac{629}{348}(x-2) - \frac{179}{232}(x-2)^2 + \frac{179}{2784}(x-2)^3 & 2 < & x < 6 \end{array}$$

2.

$$\mathbb{A} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & \frac{2}{3} & 0 & 0 \\ 0 & 4 & 30 & 0 \\ 0 & 0 & 3 & -\frac{59}{10} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{2}{3} & 0 & 0 \\ 0 & 1 & -\frac{15}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{30} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \frac{104}{177} \\ -\frac{7}{59} \\ \frac{127}{177} \\ -\frac{28}{59} \end{bmatrix}$$

3.

$$y'(x_0) \simeq \frac{-9y(x_0-h) - 8y(x_0) + 18y(x_0+h) - y(x_0+3h)}{24h}$$

$$\frac{-9 \times 0.80036477 - 8 \times 0.79704693 + 18 \times 0.79370777 - 0.78696632}{24 \times 0.01}$$

$$= -0.33285346$$

4.

$$x_1 = 1, \quad x_2 = 4, \quad x_3 = 16$$

5.

1.848920310	1.848652007	1, 848652519	1.848652518
1.848668776	1.848652511	1.848652518	
1.848653527	1.848652518		
1.848652581			

6.

$$\begin{aligned} n &= 3: \ [5.857783508, 4.410013698] \\ n &= 6: \ [4.333090144, 5.098569148, 4.916088385, 3.686536880, 1.593422050] \\ \text{Richardson} &: \ [4.845497695, 3.445377941] \end{aligned}$$

7.

$$[1.057216762, 1.165512152, 1.339654536, 1.605229535]$$

8.

$$x = 5.335298605 \quad \text{and} \quad y = 0.1857422001$$

(there appear to be three other solutions).

9.

$$\begin{bmatrix} 3 & 4.582575695 & 0 & 0 \\ 4.582575695 & 2.571428571 & 3.079446898 & 0 \\ 0 & 3.079446898 & 3.473765114 & 1.020176467 \\ 0 & 0 & 1.020176467 & -3.045193688 \end{bmatrix}$$

10.

$$5.504194286, \qquad -3.799435740 \qquad \text{and} \qquad 2.295241452$$