Introduction to **COMBINATORICS**

In **how many ways** (permutations) can we arrange n *distinct* objects in a *row*? Answer:

$$n \cdot (n-1) \cdot (n-2) \cdot ..... \cdot 3 \cdot 2 \cdot 1 \overset{def.}{=} n!$$

EXAMPLE (permuting 3 objects):

123 132 213 231 312 321

What is the number of different **permutations** of $n_1 + n_2 + ... + n_k$ objects, n_1 (n_2,...n_k) of them being '**indistinguishable**', e.g.

aabb abab abba baab baba bbaa

Another EXAMPLE: $aaabbc$. First consider them distinguishable: $a_1a_2a_3b_1b_2c$. There is 6! permutations of these. But, there is a lot of duplicity in this list, each distinct 'word' (such as $babaac$) appears 3!2!1! times: $b_1a_1b_2a_2a_3c$, $b_1a_1b_2a_3a_2c$, $b_1a_2b_2a_1a_3c$, $b_1a_2b_2a_3a_1c$, $b_1a_3b_2a_1a_1c$, $b_1a_3b_2a_2a_1c$, $b_2a_1b_1a_2a_3c$, $b_2a_1b_1a_3a_2c$, $b_2a_2b_1a_1a_3c$, $b_2a_2b_1a_3a_1c$, $b_2a_3b_1a_1a_1c$, $b_2a_3b_1a_2a_1c$.

Dividing 6! by this 'multiplicity factor' yields

$$\frac{6!}{3!2!1!} = 60$$

for the number of distinct words one can create by permuting $aaabbc$. A 'shorthand' notation for $\frac{6!}{3!2!1!}$ is $\binom{6}{3,2,1}$, called **multinomial coefficient**.

In general, the answer is

$$\frac{(n_1 + n_2 + n_3 + ... + n_k)!}{n_1!n_2!n_3!....n_k!}$$

Selecting 2 out of 3 letters (say a, b, c). There are 4 different answers, depending on:

1. The order is important, duplication is not allowed: ab, ac, ba, bc, ca, cb

2. Order irrelevant: ab, ac, bc

3. Order important again, but now each letter can be used any number of times: aa, ab, ac, ba, bb, bc, ca, cb, cc

4. Order irrelevant: aa, $ab \equiv ba$, $ac \equiv ca$, bb, $bc \equiv cb$, cc

Selecting r out of n distinct objects

1. $$n \cdot (n-1) \cdot (n-2) \cdot ..... \cdot (n-r+1) = \frac{n!}{(n-r)!} \equiv P_r^n$$

2. Each *unordered* selection of r letters will appear, in the previous list, exactly $r!$ times. The answer:
$$\frac{n!}{(n-r)!r!} \equiv C_r^n \equiv \binom{n}{r}$$
called binomial coefficient. Note the symmetry.

3. When we can use each letter repeatedly, for ordered selection we get
$$n \times n \times n \times ... \times n = n^r$$

4. The last formula is more difficult to derive. Note the each possible result can be represented by
$$\bullet\bullet \mid\mid \bullet\bullet\bullet \mid ... \mid \bullet \mid$$
with r dots and $n-1$ bars. For example, selecting 5 letters from a, b, c, d
$$\bullet\bullet \mid\mid \bullet\bullet\bullet \mid$$

represents 2 a's and 3 c's. The answer:

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

Note that the last formula also counts the number of ways to write r as a sum of n non-negative integers, order *relevant*!

Binomial expansion

$(x+y)^n = (x+y)(x+y)......(x+y) = xxx...x + yxx...x + ... + yyy...y$. This is the sum of all possible n-letter words, using a 2-letter alphabet. How many of them have exactly i x's and $n-i$ y's? Well, there is exactly $\binom{n}{i}$ such 'words' and, algebraically, they have the same 'value' of $x^i y^{n-i}$. Answer:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Example:

$$(1-3x^2)^4 = 1 + 4 \times \left(-3x^2\right) + 6 \times \left(-3x^2\right)^2 + 4 \times \left(-3x^2\right)^3 + \left(-3x^2\right)^4$$

Extension:

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + ...$$

remains true (as a Taylor expansion) even when n is negative and/or non-integer, with the understanding that $\binom{n}{i} \equiv \frac{n\cdot(n-1)\cdot...\cdot(n-i+1)}{i!}$. For example

$$\begin{aligned} (1+x)^{-3} &= 1 - 3x + 6x^2 - 10x^3 + 15x^4 - ... \\ (1+x)^{\frac{3}{2}} &= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \frac{3}{128}x^4 + ... \end{aligned}$$

Multinomial expansion

$(x+y+z)^n = (x+y+z)(x+y+z)......(x+y+z) = xxx...x + yxx...x + ... + zzz...z$ (complete 'dictionary' of all n-letter words composed of x, y and z). How many of them have i x's, j y's and k z's? Well, we know that there are $\binom{n}{i,j,k}$ of these, and each of them will be there, exactly once. Thus

$$(x+y+z)^n = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \binom{n}{i,j,k} x^i y^j z^k$$

How many terms does this formula have? We have to choose n symbols from x, y and z, repetition allowed, orderless, i.e. $\binom{n+2}{2}$.

EXAMPLES:

- $(x+y+z)^3 = x^3+y^3+z^3+3x^2y+3x^2z+3xy^2+3xz^2+3y^2z+3yz^2+6xyz$ (10 terms)

- Find the coefficient of ut^3 in the expansion of $(u+2-4t)^5$.

  Solution: The only term containing ut^3 is: $\binom{5}{1,1,3}(u)^1(2)^1(-4t)^3 = -2560ut^3$.

- $(1+2x-5x^2)^{17}$ is a 34-degree polynomial in x. When expressed as such (i.e. when expanded, and terms with like powers of x are combined), what will be the coefficient of x^4?

  Solution: $\binom{17}{i,j,k}(1)^i(2x)^j(-5x^2)^k$ is the general term of this expansion. Let us make a table of the exponents which contribute to x^4:

i	j	k
13	4	0
14	2	1
15	0	2

  . This translates to: $\binom{17}{4}(2x)^4 + \binom{17}{14,2,1}(2x)^2(-5x^2) + \binom{17}{2}(-5x^2)^2 = 680x^4$.

Partitioning

n distinct objects are to be divided into several groups of a given size each, e.g. 9 people into 'teams' of size 4, 3 and 2. Answer: $\binom{9}{4} \cdot \binom{5}{3} \cdot \binom{2}{2} = \binom{9}{4,3,2} = 1260$ - same as the number of permutations of *aaaabbbcc*.

What is some of the groups are of the same size, and we consider them 'interchangeable', e.g. divide 9 people into 3 groups of size 3 each, considering 123|456|789 and 456|123|789 as the same 'partitioning'. The answer:

$$\frac{\binom{9}{3,3,3}}{3!} = 280.$$

Another example: 20 people into 2 groups of 4, 2 groups of 3 and 3 groups of 2:

$$\frac{\binom{20}{4,4,3,3,2,2,2}}{2! \times 2! \times 3!} = 61,108,047,000$$

Circular arrangement

When arranging n people around a (circular) table (instead of a row), and we care only about who are the left and right neighbors of each person, the amount of 'duplicity' in the list of the original $n!$ (row) arrangements is n. So, in this sense, there are only $\frac{n!}{n} = (n-1)!$ distinct circular arrangements of n people.

END-OF-CHAPTER EXAMPLES:

1. A team plays a series of 10 games which they can either win (W), lose (L) or tie (T).

   (a) How many possible outcomes (order is important). Answer: $3^{10} = 59049$.

   (b) How many of these have exactly 5 wins, 4 losses and 1 tie? Answer: $\binom{10}{5,4,1} = 1260$.

   (c) Same as (a) if we *don't* care about the order of wins, losses and ties? Answer: $\binom{12}{2} = 66$ (only one of these will have 5 wins, 4 losses and 1 tie).

2. A student has to answer 20 true-false questions.

   (a) In how many distinct ways can this be done? Answer: $2^{20} = 1048576$.

   (b) How many of these will have exactly 7 correct answers? Answer: $\binom{20}{7} = 77520$.

   (c) *At least* 17 correct answers? $\binom{20}{17}+\binom{20}{18}+\binom{20}{19}+\binom{20}{20} = \binom{20}{3}+\binom{20}{2}+\binom{20}{1}+\binom{20}{0} = 1351$.

   (d) *Fewer* than 3? (*excludes* 3): $\binom{20}{0} + \binom{20}{1} + \binom{20}{2} = 211$.

3. In how many ways can 3 A's, 4 F's, 4 D's and 2 C's be arranged

   (a) in a row. Answer: $\binom{13}{3,4,4,2} = \binom{13}{3}\binom{10}{4}\binom{6}{4} = 900900$.

   (b) In how many of these will 'like' letters stay together? Answer: $4! = 24$.

   (c) Repeat (a) with circular arrangement:

   Answer: $\frac{900900}{13} = 69300$.

   (d) Repeat (b) with circular arrangement: Answer: $3! = 6$.

4. Four couples (Mr&Mrs A, Mr&Mrs B, ...) are to be seated at a round table.

   (a) In how many ways can this be done? Answer: $7! = 5040$.

   (b) How many of these have all spouses sit next to each other? Answer: $3! \times 2^4 = 96$. (Pr $= 1.905\%$).

   (c) How many of these have the men and women alternate? Answer: $4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 144$ (2.86%).

   (d) How many of these have the men (and women) sit together? Answer: $(4!)^2 = 576$ (11.43%).

5. In how many ways can we put 12 books onto 3 shelves?

(a) It the books are treated as identical: Answer: $\binom{14}{2} = 91$

(b) Books as distinct, their order important. Answer: $91 \times 12! = 43,589,145,600$.

(c) Books distinct, order irrelevant. Answer: $3^{12} = 531,441$.

6. Twelve men can be seated in a row in $12! = 479,001,600$ number of ways (trivial).

   (a) How many of these will have Mr A and Mr B sit next to each other? Answer: $2 \times 11! = 79,833,600$.

   (b) How many of the original arrangements will have Mr A and Mr B sit *apart*? Answer: $12! - 2 \times 11! = 399,168,000$.

   (c) How many of the original arrangements will have *exactly* 4 people sit *between* Mr A and Mr B? Answer: $7 \times 2 \times 10! = 50,803,200$.

7. Consider 15 distinct letters, U, R, G, F among them. These can be arranged in a row in 15! possible ways. How many of these have F and G next to each other but (at the same time) U and R apart? Answer: $2 \times 14! - 2^2 \times 13! = 149,448,499,200$.

8. Consider the standard deck of 52 cards. Deal 5 cards from this deck. This can be done in $\binom{52}{5} = 2598960$ distinct ways.

   (a) How many of these will have *exactly* 3 diamonds? Answer: $\binom{13}{3} \times \binom{39}{2} = 211,926$.

   (b) *Exactly* 2 aces? Answer: $\binom{4}{2} \times \binom{48}{3} = 103776$.

   (c) Exactly 2 aces and 2 diamonds? Answer: $\binom{1}{1}\binom{3}{1}\binom{12}{1}\binom{36}{2} + \binom{1}{0}\binom{3}{2}\binom{12}{2}\binom{36}{1} = 29,808$.

9. In how many ways can we deal 5 cards each to 4 players?

   (a) Answer: $\binom{52}{5} \times \binom{47}{5} \times \binom{42}{5} \times \binom{37}{5} = 1.4783 \times 10^{24}$

(b) So that each gets exactly one ace?

Answer: $\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} \times \binom{48}{4}\binom{44}{4}\binom{40}{4}\binom{36}{4} = 3.4127 \times 10^{21}$

(c) None gets any ace: Answer: $\binom{48}{5}\binom{43}{5}\binom{38}{5}\binom{33}{5} = 1.9636 \times 10^{23}$

(d) Mr A gets 2 *exactly* aces, the rest get none. Answer: $\binom{4}{2} \times \binom{48}{3}\binom{45}{5}\binom{40}{5}\binom{35}{5} = 2.7084 \times 10^{22}$

(e) (Any) one player gets 2 aces, the other players get none. Answer: $4 \times 2.7084 \times 10^{22} = 1.0834 \times 10^{23}$

(f) Mr. A gets 2 aces. Answer: $\binom{4}{2}\binom{48}{3} \times \binom{47}{5}\binom{42}{5}\binom{37}{5} = 5.9027 \times 10^{22}$.

(g) Mr. C gets 2 aces. Clumsy answer: $\binom{48}{5}\binom{43}{5}\times\binom{38}{3}\binom{4}{2}\times\binom{37}{5}+\binom{48}{4}\binom{4}{1}\binom{44}{5}\times\binom{39}{3}\binom{3}{2}\times\binom{37}{5} + \binom{48}{5}\binom{43}{4}\binom{4}{1} \times \binom{39}{3}\binom{3}{2} \times \binom{37}{5} + \binom{48}{4}\binom{4}{1}\binom{44}{4}\binom{3}{1} \times \binom{40}{3} \times \binom{37}{5} + \binom{48}{3}\binom{4}{2}\binom{45}{5} \times \binom{40}{3} \times \binom{37}{5} + \binom{48}{5}\binom{43}{3}\binom{4}{2} \times \binom{40}{3} \times \binom{37}{5} = 5.9027 \times 10^{22}$, or be smart and argue that ....

(h) At least one player gets 2 aces (regardless of what the others get). More difficult.

10. Roll a die five times. The number of possible (ordered) outcomes is $6^5 = 7776$. How many of these will have:

(a) One pair of identical values(and no other duplicates). Answer: $\binom{6}{1}\times\binom{5}{3}\times\binom{5}{2,1,1,1} = 3600$.

(b) Two pairs. Answer: $\binom{6}{2}\binom{4}{1} \times \binom{5}{2,2,1} = 1800$.

(c) A triplet: $\binom{6}{1}\binom{5}{2} \times \binom{5}{3,1,1} = 1200$.

(d) 'Full house' (a triplet and a pair): $\binom{6}{1}\binom{5}{1} \times \binom{5}{3,2} = 300$.

(e) 'Four of a kind': $\binom{6}{1}\binom{5}{1} \times \binom{5}{4,1} = 150$.

(f) 'Five of a kind': $\binom{6}{1} \times \binom{5}{5} = 6$.

(g) Nothing. $6 \times 5 \times 4 \times 3 \times 2 = 720$.

Note that all these answers properly add up to 7776 (check).

11. Let us try the same thing with 15 rolls of a die ($6^{15} = 4.7018 \times 10^{11}$ outcomes in total).

    How many of these will have:

    (a) A quadruplet, 2 triplets, 2 pairs and 1 singlet: $\binom{6}{1}\binom{5}{2}\binom{3}{2}\binom{1}{1} \times \binom{15}{4,3,3,2,2,1} = 6.8108 \times 10^{10}$

    (b) 3 triplets and 3 pairs: $\binom{6}{3}\binom{3}{3} \times \binom{15}{3,3,3,2,2,2} = 1.5135 \times 10^{10}$.

    We will not try to complete this exercise; the full list would consist of 110 possibilities.