RANDOM EXPERIMENTS

Definition: For a specific random experiment, its **sample space** consists of a list (set) of all possible outcomes. The individual (complete) outcomes are called **simple events**. If we are lucky, all simple events are `equally likely`.

A few EXAMPLES:

1. Rolling a die

   $\{1, 2, 3, 4, 5, 6\}$

2. Rolling a die n times (rolling n dice at once). The sample space is the set of all 6^n n-digit numbers, made up of 1, 2, ...6. For $n = 2$, this yields

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

   Note that the order is important (12 and 21 are considered different). This makes the simple events equally likely.

3. Selecting k objects out of n (2 people out of 4, 5 cards out of 52, etc.). Here, the SS consists of C^n_k unordered sections (all equally likely), such as $\{AB, AC, AD, BC, BD, CD\}$.

4. Flipping a coin until a head appears. This time, the SS is (countably) infinite: $\{H, TH, TTH, TTTH, TTTTH, TTTTTH, ...\}$, which implies that the outcomes can no longer be equally likely.

5. Rotating a wheel with a pointer. The SS consists of all angles (at which the pointer stops) from 0 to 2π (in radians). The set of all such real numbers is infinity, yet they are still equally likely. This implies that the probability of any of them is 0, and we have to introduce the concept of `probability density function` instead (second half of the course).

6. Flipping a tack ($\perp$). The sample space is very simple: $\{\perp, \angle\}$, but how do we assign probabilities? This can be done only empirically (based on the observed frequencies of occurrence) - until this is done, we have to assume that the probabilities are p and $1 - p$.

An **event** is a *subset* of the sample space, usually denoted by a capital letter from the beginning of the alphabet.

EXAMPLES (rolling a die twice):

1. A: the total number of dots equals 8.
2. B: neither of the two numbers is a six.
3. C: first number smaller than second.

Thus, we need a few basics of **Set Theory**, even though we may occasionally use different terminology: Sample Space instead of Universal Set, Simple Event instead of Element, Event instead of Subset, and Null Event instead of Empty Set. But we do use the same `intersection` (notation: $A \cap B$, also A and B), `union` $A \cup B$ (A or B - this is the non-exclusive or) and `complement` $\overline{A}$ (not A). When dealing with 2 or 3 events, a lot of useful information can be gathered from a Venn diagram, but beyond that, we need the full power of **Boolean Algebra**:

Both $\cap$ and $\cup$ (individually) are *commutative* and *associative*.

Intersection is *distributive* over union: $A \cap (B \cup C \cup ...) = (A \cap B) \cup (A \cap C) \cup ...$

Similarly, union is distributive over intersection: $A \cup (B \cap C \cap ...) = (A \cup B) \cap (A \cup C) \cap ...$

A few trivial rules: $A \cap \Omega = A$, $A \cap \emptyset = \emptyset$, $A \cap A = A$, $A \cup \Omega = \Omega$, $A \cup \emptyset = A$, $A \cup A = A$, $A \cap \overline{A} = \emptyset$, $A \cup \overline{A} = \Omega$, $\overline{\overline{A}} = A$.

Also, $A \subset B$ implies that $A \cap B = A$ and $A \cup B = B$.

`De Morgan's Laws`: $\overline{A \cap B \cap C \cap ...} = \overline{A} \cup \overline{B} \cup \overline{C} \cup ...$, and $\overline{A \cup B \cup C \cup ...} = \overline{A} \cap \overline{B} \cap \overline{C} \cap ...$

A and B are called (mutually) `exclusive` (or disjoint) when $A \cap B = \emptyset$.

Rules of probability

`Probability` of a *simple* event is defined as the relative frequency of its occurrence in an 'infinite' independent repetition of the corresponding experiment. In many cases, it can be established based on the symmetry of the experiment.

Probability of any other event is the *sum* of probabilities of the simple events it consists of.

This immediately implies:

$$\begin{aligned} \Pr(A) &\geq 0 \\ \Pr(\emptyset) &= 0 \\ \Pr(\Omega) &= 1 \end{aligned}$$

Note that $\Pr(A) = 0$ does not necessarily imply that $A = \emptyset$, some non-empty events may have a zero probability (we have already seen an example).

A less trivial formula is

$$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$

when $A \cap B = \emptyset$. This implies that $\Pr(\overline{A}) = 1 - \Pr(A)$ and $\Pr(A \cap \overline{B}) = \Pr(A) - \Pr(A \cap B)$ as two special cases.

When A and B are not exclusive, we have to use

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

which can be extended to $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$ and, in general:

$$\begin{aligned}
&\Pr(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k) = \\
&\sum_{i=1}^{k} \Pr(A_i) - \sum_{i<j}^{k} \Pr(A_i \cap A_j) + \sum_{i<j<\ell}^{k} \Pr(A_i \cap A_j \cap A_\ell) \\
&-\ldots \pm \Pr(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_k)
\end{aligned}$$

(the plus sign for k odd, the minus sign for k even). The formula computes the probability that *at least one* of the A_i events happens.

The probability of getting *exactly one* of the A_i events would be similarly computed by:

$$\begin{aligned}
&\sum_{i=1}^{k} \Pr(A_i) - 2\sum_{i<j}^{k} \Pr(A_i \cap A_j) + 3\sum_{i<j<\ell}^{k} \Pr(A_i \cap A_j \cap A_\ell) \\
&-\ldots \pm k \Pr(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_k)
\end{aligned}$$

EXAMPLE:

The first k integers are 'shuffled'. What is the probability that they all end up in their 'natural' place? Answer: $\frac{1}{k!}$. At least one number will be misplaced: $1 - \frac{1}{k!}$ (complement). All will be misplaced:

$$\Pr(\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_k}) =$$

$$\Pr(\overline{A_1 \cup A_2 \cup ... \cup A_k}) =$$

$$1 - \Pr(A_1 \cup A_2 \cup ... \cup A_k) =$$

$$1 - \sum_{i=1}^{k} \Pr(A_i) + \sum_{i<j}^{k} \Pr(A_i \cap A_j) -$$

$$... \mp \Pr(A_1 \cap A_2 \cap ... \cap A_k) =$$

$$1 - k \cdot \frac{1}{k} + \binom{k}{2} \cdot \frac{1}{k(k-1)}$$

$$-\binom{k}{3} \frac{1}{k(k-1)(k-2)} + ... \mp \frac{1}{k!} =$$

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + ... \mp \frac{1}{k!} \equiv Q_k$$

Exactly i of them will end up in their original position (implying $k - i$ will be misplaced):

$$\tbinom{k}{i} \times \frac{1}{P_i^k} \times Q_{k-i} = \frac{k!}{i!(k-i)!} \times \frac{(k-i)!}{k!} \times Q_{k-i} = \frac{Q_{k-i}}{i!}$$

Thus, for example, when $k = 5$, the probability of 0, 1, 2, 3, 4 and 5 matches is $\frac{11}{30}$, $\frac{3}{8}$, $\frac{1}{6}$, $\frac{1}{12}$, 0 and $\frac{1}{120}$ respectively. ■

The **main point** of these rules is: Probability of any (Boolean) expression involving A, B, C, ... can be *always* converted to probabilities involving the individual events and their *simple* (non-complemented) *intersections*.

EXAMPLES:

$$\Pr[(A \cap B) \cup \overline{B \cap C}] =$$

$$\Pr(A \cap B) + \Pr(\overline{B \cap C}) - \Pr(A \cap B \cap \overline{B \cap C}) =$$

$$\Pr(A \cap B) + 1 - \Pr(B \cap C) - \Pr(A \cap B)$$

$$+ \Pr(A \cap B \cap B \cap C) = 1 - \Pr(B \cap C) + \Pr(A \cap B \cap C)$$

■

$$\Pr[(A \cap B) \cup \overline{C \cup D}] = \Pr(A \cap B) + \Pr(\overline{C \cup D})$$

$$- \Pr(A \cap B \cap \overline{C \cup D}) = \Pr(A \cap B) + 1 - \Pr(C \cup D)$$

$$- \Pr(A \cap B) + \Pr[A \cap B \cap (C \cup D)] = 1 - \Pr(C \cup D)$$

$$+ \Pr[(A \cap B \cap C) \cup (A \cap B \cap D)] = 1 - \Pr(C) - \Pr(D)$$

$$+ \Pr(C \cap D) + \Pr(A \cap B \cap C) + \Pr(A \cap B \cap D)$$

$$- \Pr(A \cap B \cap C \cap D)$$

■

Back to: Four players are dealt 5 cards each. What is the probability that *at least* one player gets *exactly* 2 aces?

$$\Pr(A_1 \cup A_2 \cup A_3 \cup A_4) =$$

$$\sum_{i=1}^{4} \Pr(A_i) - \sum_{i<j}^{4} \Pr(A_i \cap A_j) + 0 - 0 =$$

$$4 \times \frac{\binom{4}{2} \times \binom{48}{3}}{\binom{52}{5}} - 6 \times \frac{\binom{4}{2,2,0} \times \binom{48}{3,3,42}}{\binom{52}{5,5,42}} = 15.75\%$$

■

There are 100,000 lottery tickets marked 00000 to 99999. One is selected at random. What is the probability that the number on it contains 84 (consecutive, in that order) *at least* once?

Let's introduce four events: A means that the first two digits of the ticket are 84, B: 84 is found in the second and third position, C: 84 in position three and four, and D: 84 in the last two positions.

$$\Pr(A \cup B \cup C \cup D) =$$

$$\Pr(A) + \Pr(B) + \Pr(C) + \Pr(D)$$

$$- \Pr(A \cap C) - \Pr(A \cap D) - \Pr(B \cap D) + 0 =$$

$$4 \times \frac{1,000}{100,000} - 3 \times \frac{10}{100,000} = 0.04 - 0.0003 = 3.97\%$$

■

Tom, Frank Jim and 5 other boys are randomly seated at a round table. What is the probability that neither Tom nor Frank will sit next to Jim?

Now we introduce: A: Tom sits next to Jim, and B: Frank sits next to Jim. Then, clearly

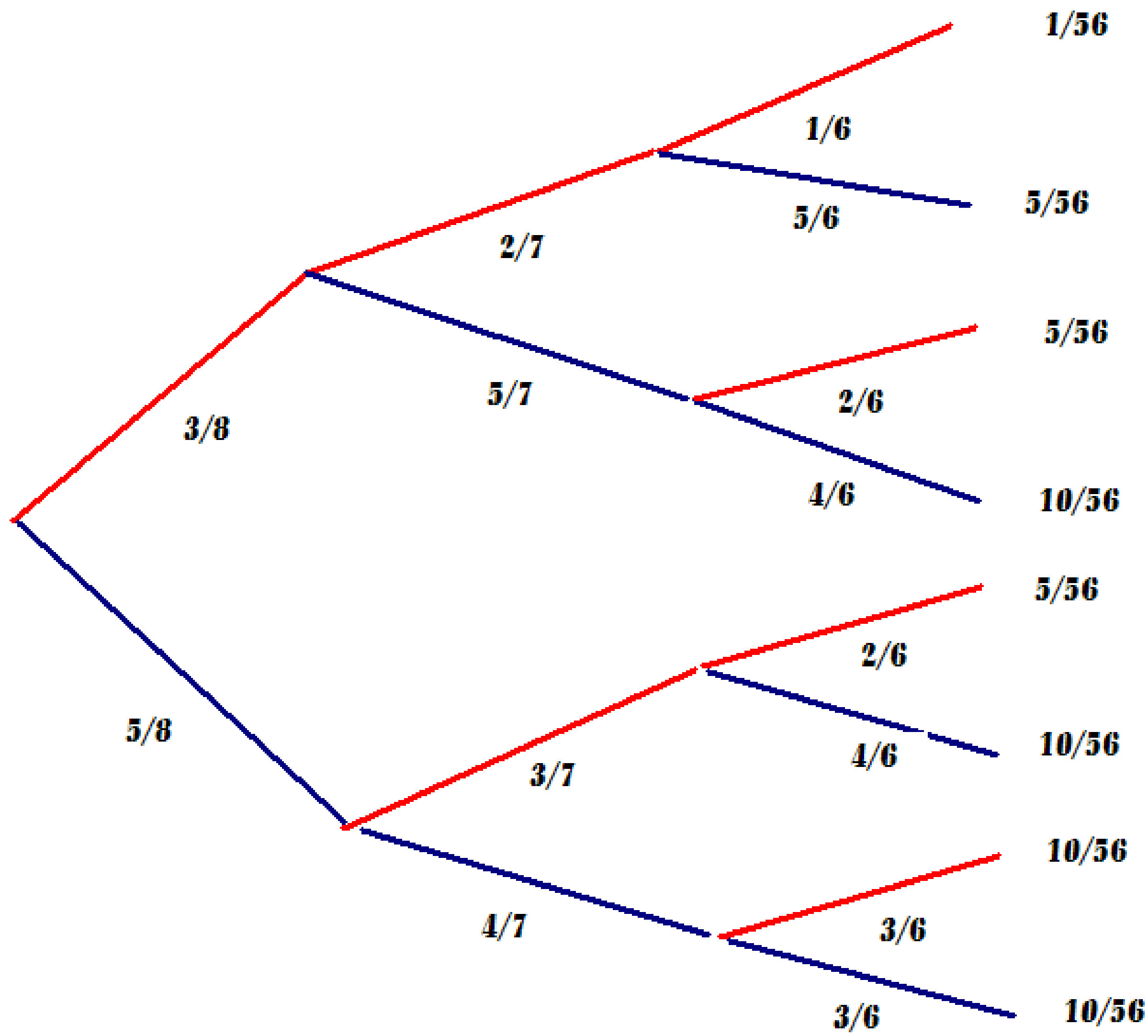
$$\Pr(\bar{A} \cap \bar{B}) = \Pr(\overline{A \cup B}) = 1 - \Pr(A \cup B) =$$

$$1 - \Pr(A) - \Pr(B) + \Pr(A \cap B) =$$

$$1 - \frac{2 \times 6!}{7!} - \frac{2 \times 6!}{7!} + \frac{2 \times 5!}{7!} = \frac{10}{21} = 47.62\%$$

Probability tree and **conditional probabilities**

Consider a random experiment which is done in several **stages** such as, for example, selecting 3 marbles (one by one, without replacement), from a box containing (originally) 3 red and 5 blue marbles. The easiest way to display possible outcomes of this experiment is to draw a so called **probability tree** (a graphical representation of a sample space):



Each `path` through this graph represents one simple event.

It is easy to assign probabilities to individual `branches` of this tree (note that the probabilities at each 'fork' have to add up to one).

Let us try to answer a few questions:

Which subset represents the event R_1 (first marble red); B_2 (second marble blue), etc.

What are the probabilities of individual branches? $\frac{3}{8}$ is obviously $\Pr(R_1)$. But how about

$\frac{5}{7}$; it's definitely *not* $\Pr(B_2)$.To give it its proper meaning, we have to introduce the notion of **conditional probability** of an event (B_2, in this case) **given** that another event (R_1) has already happened. The notation: $\Pr(B_2|R_1)$. Similarly $\frac{2}{6}$ is $\Pr(R_3|R_1 \cap B_2)$.

How do we compute probabilities of simple events (and thus events in general)? Clearly, the probability that the corresponding random experiment will take the *rbr* path is $\frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} = \frac{5}{56}$, etc. Formally, this reads:

$$\Pr(R_1 \cap B_2 \cap R_3) = \Pr(R_1) \cdot \Pr(B_2|R_1) \cdot \Pr(R_3|R_1 \cap B_2)$$

This is the so called **product rule**, and can be generalized to any number of events:

$$\begin{aligned} \Pr(A \cap B) &= \Pr(A) \cdot \Pr(B|A) \\ \Pr(A \cap B \cap C \cap D) &= \Pr(A) \cdot \Pr(B|A) \cdot \Pr(C|A \cap B) \\ &\quad \cdot \Pr(D|A \cap B \cap C) \\ &\quad \vdots \end{aligned}$$

■

EXAMPLE: 4 players are dealt 13 cards each from an ordinary deck (of 52 cards). What is the probability that each player will get exactly one ace? Utilizing the product rule:

$$\frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} \cdot \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}} \cdot \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} \cdot \frac{\binom{1}{1}\binom{12}{12}}{\binom{13}{13}} = 10.55\%$$

■

In general (i.e. for *any* random experiment, and *any* two events), we *define* the conditional probability of B given A by:

$$\Pr(B|A) \equiv \frac{\Pr(A \cap B)}{\Pr(A)}$$

■

EXAMPLE: The experiment consists of rolling two dice (red and green), A is: 'the total number of dots equals 6' ($\bigcirc$), B is: 'the red die shows an even number' ($\times$). Compute $\Pr(B|A)$.

$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bigcirc$	$\bullet$
$\times$	$\times$	$\times$	$\otimes$	$\times$	$\times$
$\bullet$	$\bullet$	$\bigcirc$	$\bullet$	$\bullet$	$\bullet$
$\times$	$\otimes$	$\times$	$\times$	$\times$	$\times$
$\bigcirc$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

Answer: $\frac{2/36}{5/36} = \frac{2}{5}$. Note that $\Pr(A|B) = \frac{2/36}{18/36} = \frac{1}{9}$ is different (there is no reason to expect them to be related in any way).

The **meaning** of any such conditional probability is as follows: In many such cases we have to assume that the (whole) experiment has already happened, but we get only an incomplete information about its outcome (someone who observed it all can tell us that A has happened, but refuses to tell us anything else). This means that the whole sample space has shrunk to A, and the probabilities of its simple events have to be readjusted accordingly.

Using this general definition of conditional probability, we can now compute $\Pr(B_1|R_3) = \frac{5+10}{1+5+5+10} = \frac{5}{7}$.

Note that all formulas of probability are valid if we (consistently) insert the same condition, e.g.

$$\begin{aligned}\Pr(\overline{B}|A) &= 1 - \Pr(B|A) \\ \Pr(A \cup B|C) &= \Pr(A|C) + \Pr(B|C) - \Pr(A \cap B|C)\end{aligned}$$

■

Definition: `Partition` of a sample space is a collection of k (any integer) events, say A_1, A_2, A_3, ..., A_k, so that they (i) are mutually exclusive (no 'overlaps'): $A_i \cap A_j = \emptyset$ for any i and j, and (ii) cover the whole sample space (no 'gaps'): $A_1 \cup A_2 \cup A_3 \cup ... \cup A_k = \Omega$.

Obviously, the 'finest' partition is the collection of all simple events, and the 'crudest' partition is Ω itself. The most interesting partitions will of course be the in-between cases. A common example is A and $\overline{A}$ (where A is an arbitrary event).

Partitions (introduced by *us*) come useful when computing probability of another event B, by utilizing the **Formula of Total Probability**:

$$\begin{aligned}\Pr(B) &= \Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2) + ... \\ &\quad + \Pr(A_k)\Pr(B|A_k)\end{aligned}$$

■

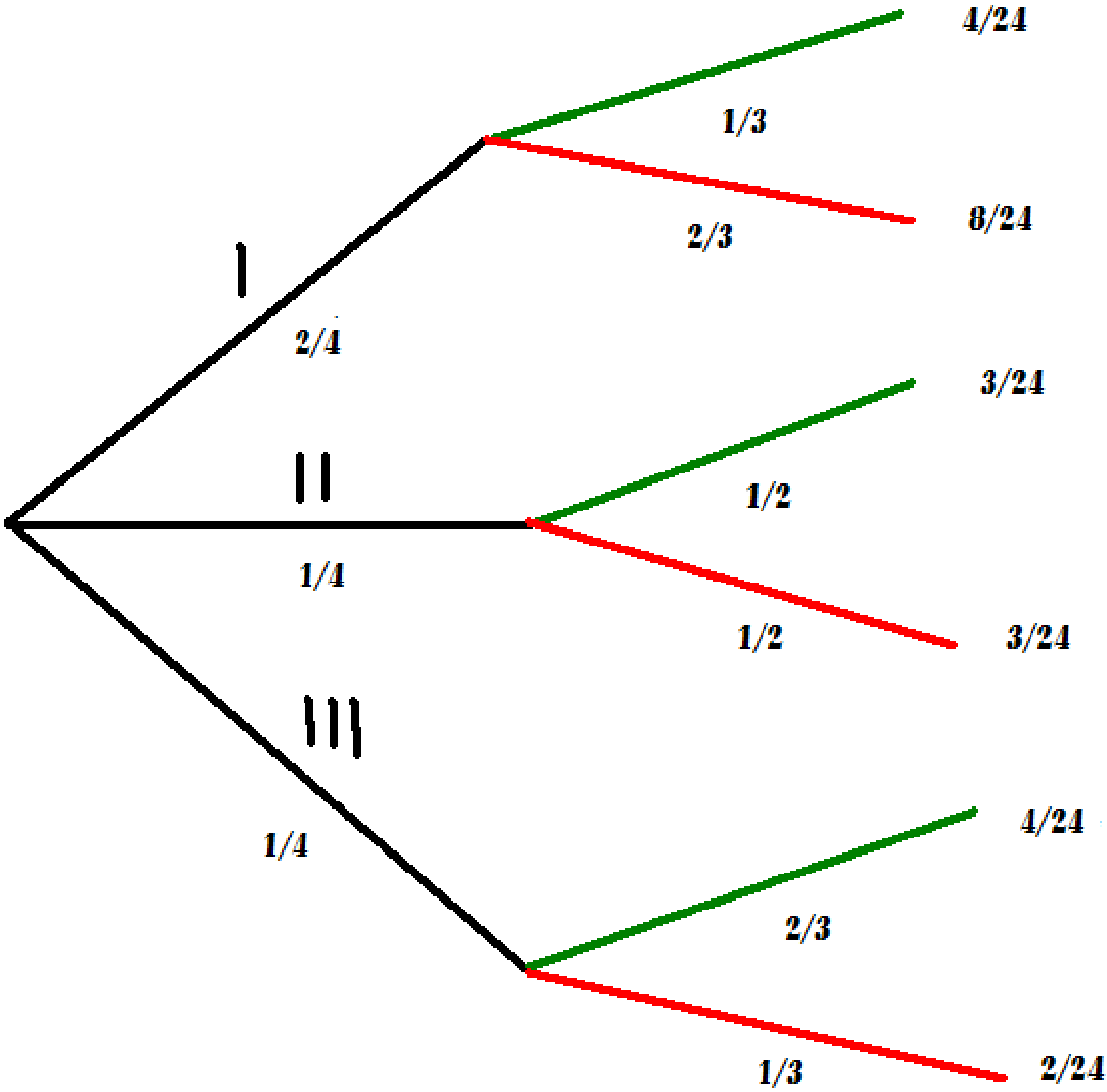
EXAMPLE: Two players are dealt 5 cards each. What is the probability that they will have the same number of aces?

We partition the sample space according to how many aces the first player gets, calling the events A_0, A_1, ..., A_4.(using a convenient index) Answer: $\frac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}} \cdot \frac{\binom{4}{0}\binom{43}{5}}{\binom{47}{5}} + \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} \cdot \frac{\binom{3}{1}\binom{44}{4}}{\binom{47}{5}} + \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} \cdot \frac{\binom{2}{2}\binom{45}{3}}{\binom{47}{5}} + \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} \cdot 0 + \frac{\binom{4}{4}\binom{48}{0}}{\binom{52}{5}} \cdot 0 = 49.33\%$

bBayes' rule

Let's start with an EXAMPLE:

Consider four 'black' boxes: two of them (Type I) have 1 green and 2 red marbles inside, one (Type II) has 1 green and 1 red marble, and one (Type III) has 2 green and 1 red marble. One of these boxes is selected at random, and a marble drawn from it.



Compute $\Pr(R)$:

$$\Pr(R|I) \cdot \Pr(I) + \Pr(R|II) \cdot \Pr(II) + \Pr(R|III) \cdot \Pr(III)$$

$= \frac{8}{24} + \frac{3}{24} + \frac{2}{24} = 54.17\%$. Similarly, $\Pr(G) = \frac{11}{24}$.

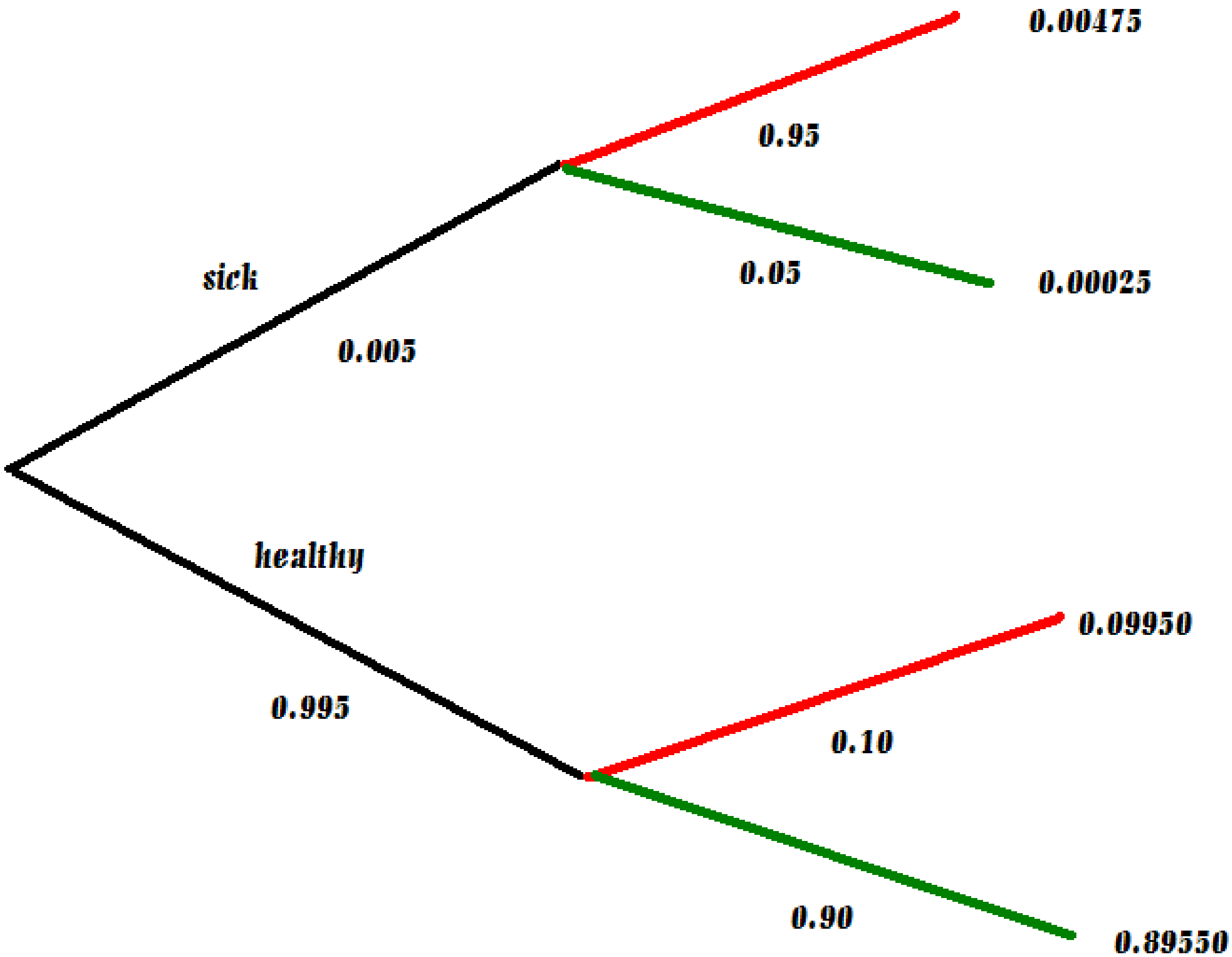
How about $\Pr(\mathrm{I}\,|R)$:

$$\begin{aligned}\Pr(I|R) &= \frac{\Pr(\mathrm{I}\cap R)}{\Pr(R)} = \\ \frac{8/24}{13/24} &= \frac{8}{13} = 61.54\% > 50\%\end{aligned}$$

The procedure for computing $\Pr(\mathrm{I}\,|R)$ can be generalized as follows: check off ($\checkmark$) all simple events contributing to R, out of these check off (using a different symbol, $\bigcirc$ in our case) those which also contribute to I Then divide the probability of the latter set by the total probability of the former.

Another EXAMPLE:

Let 0.5% of a population in a certain area have TB. There is a medical test which can detect this condition in 95% of all (infected) cases, but at the same time the test is (falsely) positive for 10% of the healthy people. A person is selected randomly and tested. The test is positive. What is the probability that the person actually has it?



Using the Bayes' rule, this is equal to $\frac{0.00475}{0.00475+0.09950} = 4.556\%$ (almost 10 times bigger than before the test)!

Independence

of *two* events is a very natural notion: if the experiment is done in such a manner that A (happening or not) cannot influence the probability of B, B is independent of A. Formally, this means that $\Pr(B|A) = \Pr(B)$. Mathematically, this is equivalent to $\Pr(A \cap B) =$

$\Pr(A) \cdot \Pr(B)$, and also to $\Pr(A|B) = \Pr(A)$, meaning that A is independent of B. The same condition is also equivalent to $\Pr(A \cap \overline{B}) = \Pr(A) \cdot \Pr(\overline{B})$, etc.

Example: An outcome of one die cannot influence the outcome of another die; but also: an outcome of a die cannot influence any of its future outcomes (no 'memory').

I should mention that the condition of independence may sometimes be met 'accidently' (not of much use to us).

Don't confuse with 2 events being exclusive!!

Three events are **mutually** independent if they are pairwise independent, plus:

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

(the natural independence is of the mutual type). Mutual independence of A, B and C implies that of A, B and $\bar{C}$, but also this: any event build from A and B (e.g. $A \cup \overline{B}$) must be independent of C.

Mutual independence of k events means that the probability of intersection of *any number* (i.e. 2, 3, ...) of these equals the *product* of the individual probabilities. This implies that any event build out of A, B, ... must be independent of any event build out of C, D, ... ,as long as the two sets are *distinct*.

The most important implication of independence is this: the probability of a Boolean expression involving independent events can be expressed in terms of the *individual* probabilities.

EXAMPLES:

$$\Pr[(A \cup B) \cap \overline{C \cup D}] =$$

$$\Pr(A \cup B) \cdot [1 - \Pr(C \cup D)] =$$

$$[\Pr(A) + \Pr(B) - \Pr(A)\Pr(B)] \cdot$$

$$[1 - \Pr(C) - \Pr(D) + \Pr(C)\Pr(D)]$$

■

$$\Pr[(A \cup \overline{C \cap D}) \cap \overline{B \cup \overline{D}}] =$$

$$\Pr[(A \cup \overline{C} \cup \overline{D}) \cap \overline{B} \cap D] =$$

$$\Pr[(A \cup \overline{C} \cup \overline{D}) \cap D] \cdot [1 - \Pr(B)] =$$

$$\Pr[(A \cap D) \cup (\overline{C} \cap D) \cup \emptyset] \cdot [1 - \Pr(B)] =$$

$$\{\Pr(A)\Pr(D) + \Pr(D) - \Pr(C)\Pr(D) - \Pr(A) \cdot$$

$$[1 - \Pr(C)]\Pr(D)\} \cdot [1 - \Pr(B)]$$

To make it easier, I usually give you specific values for $\Pr(A)$, $\Pr(B), ...$

END-OF-CHAPTER EXAMPLES:

- Let us return to the lottery with 100,000 tickets, and compute the probability that a randomly selected ticket has an 8 and a 4 on it (each at least once, in any order, and not necessarily consecutive). This is a roll-of-a-die type of experiment!

  Define A: no 8 at any place, B: no 4. We need

$$\Pr(\overline{A} \cap \overline{B}) = \Pr(\overline{A \cup B}) = 1 - \Pr(A \cup B)$$

$$= 1 - \Pr(A) - \Pr(B) + \Pr(A \cap B)$$

Clearly $A \equiv A_1 \cap A_2 \cap ... \cap A_5$, where A_1: 'no 8 in the first place', A_2: 'no 8 in the second place', etc. A_1, A_2, ..., A_5 are mutually independent, thus $\Pr(A) = \Pr(A_1) \cdot \Pr(A_2) \cdot ... \cdot \Pr(A_5) = (\frac{9}{10})^5$. Similarly, $\Pr(B) = (\frac{9}{10})^5$. Now, $A \cap B \equiv C_1 \cap C_2 \cap ... \cap C_5$ where C_1: not 8 nor 4 in the first spot, C_2: not 8 nor 4 in the second, etc. These of course are also independent, which implies $\Pr(A \cap B) = (\frac{8}{10})^5$.

Answer: $1 - 2(\frac{9}{10})^5 + (\frac{8}{10})^5 = 14.67\%$.

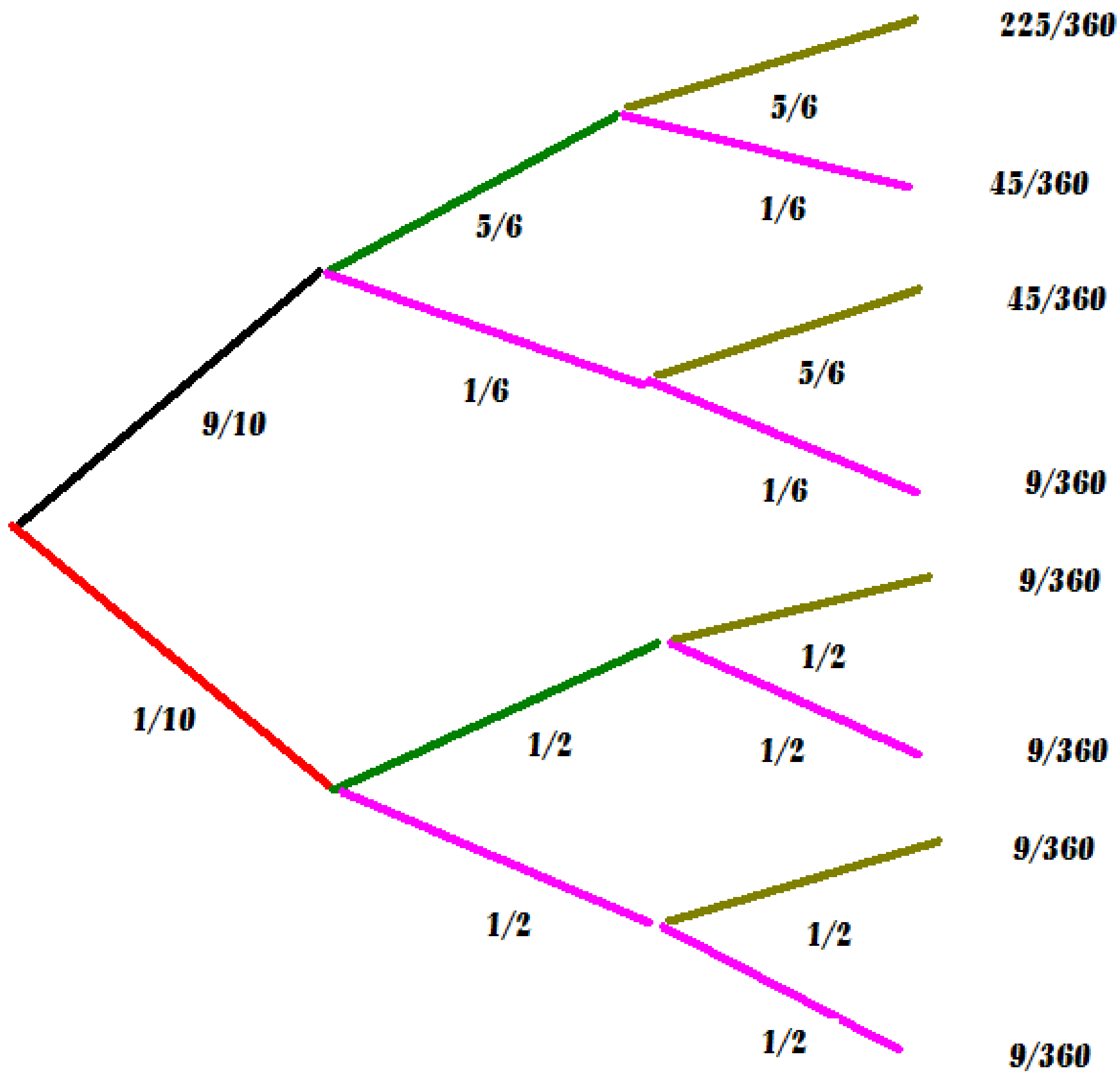
- The same question, but this time we want *at least* one 8 *followed* (sooner or later) by a 4 (at least once). What makes this different from the original question is that 8 and 4 now don't have to be consecutive.

  We *partition* the sample space according to the *position* at which 8 appears *for the first time*: B_1, B_2, ..., B_5, plus B_0 (which means there is no 8). Now, if A is the event of our question (8 followed by a 4), we apply the formula of total probability:

$$\Pr(A) = \Pr(B_1)\Pr(A|B_1) + ... + \Pr(B_5)\Pr(A|B_5)$$

$$+\Pr(B_0)\Pr(A|B_0) = \frac{1}{10} \cdot [1 - (\frac{9}{10})^4] +$$
$$\frac{9}{10} \cdot \frac{1}{10} \cdot [1 - (\frac{9}{10})^3] + (\frac{9}{10})^2 \cdot \frac{1}{10} \cdot [1 - (\frac{9}{10})^2] +$$
$$(\frac{9}{10})^3 \cdot \frac{1}{10} \cdot [1 - \frac{9}{10}] + 0 + 0 = 8.146\%$$

- Out of 10 dice, 9 of which are regular but one is 'crooked' (6 has a probability of 0.5), a die is selected at random and rolled twice.



Given that the first roll resulted in a six, what is the (conditional) probability of getting a six again in the second roll?

$$\frac{9+9}{9+9+45+9} = 25\%$$

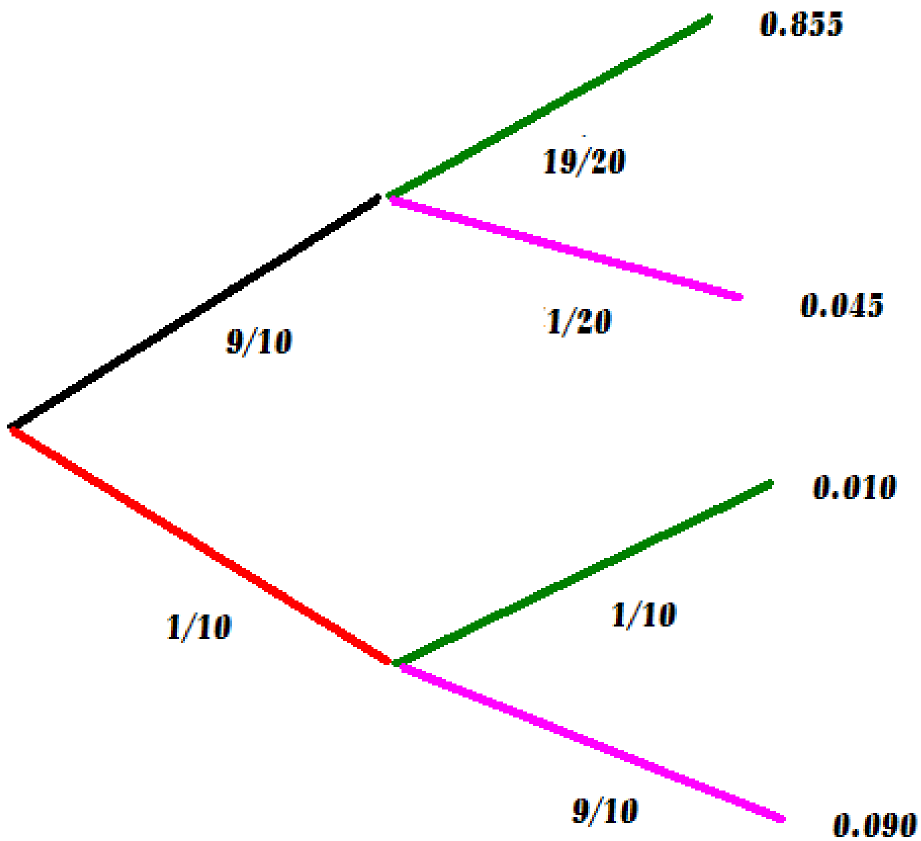
Are S_1 and S_2 independent? We have to check: $\Pr(S_1 \cap S_2) \stackrel{?}{=} \Pr(S_1) \cdot \Pr(S_2)$.

$$(\frac{1}{20} =) \;\; \frac{18}{360} \neq \frac{72}{360} \cdot \frac{72}{360} \;\; (= \frac{1}{25})$$

Given that both rolls resulted in a six, what is the (conditional) probability of having selected the crooked die?

$$\frac{9}{9+9} = 50\%$$

- Ten people have been arrested as suspects in a crime one of them must have committed. A lie detector will (incorrectly) incriminate an innocent person with a 5% probability, it can (correctly) detect a guilty person with a 90% probability.



One person has been randomly selected and tested; the lie detector has its red light flashing. What is the probability that he is the criminal?

$$\frac{0.090}{0.090+0.045} = \frac{2}{3}$$

All 10 people have been tested and exactly one incriminated. What is the probability of having the criminal now?

The conditional sample space consists of 10 simple events (out of the original $2^{10} = 1024$):

$$\begin{array}{ll} rggggggggg & 0.9 \times 0.95^9 \\ grgggggggg & 0.1 \times 0.95^8 \times 0.05 \\ \ldots\ldots\ldots & \\ gggggggggr & 0.1 \times 0.95^8 \times 0.05 \end{array}$$

Answer: $\dfrac{0.9 \times 0.95^9}{0.9 \times 0.95^9 + 9 \times 0.1 \times 0.95^8 \times 0.05} = 95\%$.

- Two men take one shot each at a target. Mr. A can hit it with the probability of $\frac{1}{4}$, Mr. B's chances are $\frac{2}{5}$. What is the probability that the target is hit (at least once)?

  Here, we have to, *on our own*, assume independence of the two shots.

  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \frac{1}{4} + \frac{2}{5} - \frac{1}{10} = 55\%$, or: $\Pr(A \cup B) = 1 - \Pr(\overline{A \cup B}) = 1 - \Pr(\overline{A} \cap \overline{B}) = 1 - \frac{3}{4} \cdot \frac{3}{5} = 55\%$.

- What is more likely, getting *at least one* 6 in `four` rolls of a die, or getting at least one *double 6* in `twenty four` rolls of a *pair* of dice?

  The first probability is $1 - (\frac{5}{6})^4 = 51.77\%$. The second one, similarly, equals $1 - (\frac{35}{36})^{24} = 49.14\%$.

- Four people are dealt 13 cards each. You (one of the players) got one ace. What is the probability that your partner has the other three aces?

  This is a very simple (natural) conditional probability, if we assume that we are the first player to be dealt his 13 cards, and our partner is the next one: $\frac{\binom{3}{3}\binom{36}{10}}{\binom{39}{13}} = 3.129\%$.

- A, B, C are mutually independent, having (the individual) probabilities of 0.25, 0.35 and 0.45, respectively. Compute $\Pr[(A \cap \overline{B}) \cup C]$.

$$\Pr(A \cap \overline{B}) + \Pr(C) - \Pr(A \cap \overline{B} \cap C) =$$

$$0.25 \times 0.65 + 0.45 - 0.25 \times 0.65 \times 0.45 = 53.94\%$$

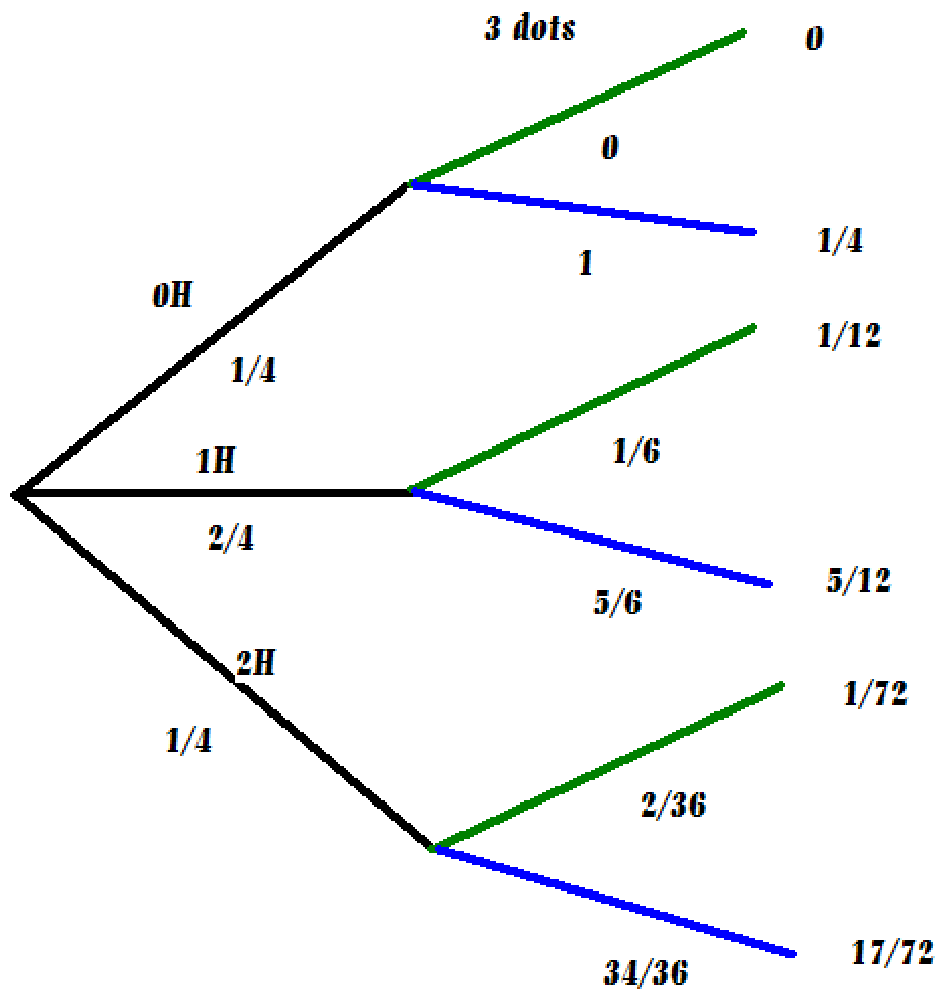
- Two coins are flipped, followed by rolling a die as many times as the number of heads shown. What is the probability of getting fewer than 5 dots in total?

  Introduce a partition A_0, A_1, A_2 according to how many heads are obtained. If B stands for 'getting fewer than 5 dots', the total-probability formula yields:

$$\Pr(B) = \Pr(A_0)\Pr(B|A_0) + \Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2)$$

$$= \frac{1}{4} \times 1 + \frac{2}{4} \times \frac{4}{6} + \frac{1}{4} \times \frac{6}{36} = 62.5\%$$

  Given that there were exactly 3 dots in total, what is the conditional probability that the coins showed exactly one head?



Answer: $\frac{1/12}{1/12+1/72} = 85.71\%$.

- Jim, Joe, Tom and six other boys are randomly seated in a row. What is the probability that *at least two* of the three friends will sit next to each other?

  Let's introduce A: Jim and Joe sit together, B: Jim and Tom sit together, C: Joe and Tom sit together.

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C)$$

$$- \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$$

There is 9! random arrangements of the boys, $2 \times 8!$ will meet condition A (same with B and C), $2 \times 7!$ will meet both A and B (same for $A \cap C$ and $B \cap C$), *none* will meet all three. Answer: $3 \times \frac{2\times 8!}{9!} - 3 \times \frac{2\times 7!}{9!} = 58.33\%$ (try this with *circular* arrangement).

- There are 10 people at a party (no twins). Assuming that all 365 days of a year are equally likely to be someone's birth date, and also ignoring leap years, what is the probability of all these ten people having different birth dates? Hint: This is a roll-of-a-die type of experiment.

$$\frac{P_{10}^{365}}{365^{10}} = 88.31\%$$

Exactly *two* people having the same birth date (the rest all distinct)?

$$\frac{365 \times \binom{10}{2} \times P_8^{364}}{365^{10}} = 11.16\%$$

These two answers account for 99.47% of the total probability. Two or three duplicates, and perhaps one triplicate would most likely take care of the rest; try it!