IIt is easy to derive a general formula for the probability of intersection of many events, some of them with bars. How it looks like should be clear from the following example:

$$
\begin{aligned}
&\Pr\left(A_1 \cap A_2 \cap \overline{B_1} \cap \overline{B_2} \cap \overline{B_3}\right) = \Pr\left(A_1 \cap A_2\right) \qquad (1)\\
&- \Pr\left(A_1 \cap A_2 \cap B_1\right) - \Pr\left(A_1 \cap A_2 \cap B_2\right) - \Pr\left(A_1 \cap A_2 \cap B_3\right)\\
&+ \Pr\left(A_1 \cap A_2 \cap B_1 \cap B_2\right) + \Pr\left(A_1 \cap A_2 \cap B_1 \cap B_3\right) + \Pr\left(A_1 \cap A_2 \cap B_2 \cap B_3\right)\\
&- \Pr\left(A_1 \cap A_2 \cap B_1 \cap B_2 \cap B_3\right)
\end{aligned}
$$

Note that we start with the probability of the intersection of all unbarred events, than extend this intersection, one by one, by each of the barred events (removing the bar), every possible combination of two barred events (without bars), three barred events, etc., letting the signs alternate.

Proof.

$$
\begin{aligned}
&\Pr\left(A_1 \cap A_2 \cap \overline{B_1 \cup B_2 \cup B_3}\right) = \Pr\left(A_1 \cap A_2\right) - \Pr\left(A_1 \cap A_2 \cap (B_1 \cup B_2 \cup B_3)\right)\\
&= \Pr\left(A_1 \cap A_2\right) - \Pr\left((A_1 \cap A_2 \cap B_1) \cup (A_1 \cap A_2 \cap B_2) \cup (A_1 \cap A_2 \cap B_3)\right)
\end{aligned}
$$

where the second term is then expanded in the usual manner of (3), which was proved earlier. ■

Based on this rule, we can now derive the following formulas for the probability of getting exactly 1 (2, 3, ...) events out of k (we will use $k = 5$ in our example, and call the events A_1, A_2, ..., A_5):

$$
\begin{aligned}
&\Pr(\text{exactly 1 of the 5 } A\text{s}) = \sum_{i=1}^{5} \Pr(A_i) - 2\sum_{i<j}^{5} \Pr(A_i \cap A_j) + 3\sum_{i<j<k}^{5} \Pr(A_i \cap A_j \cap A_k)\\
&-4\sum_{i<j<k<\ell}^{5} \Pr(A_i \cap A_j \cap A_k \cap A_\ell) + 5\Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)
\end{aligned}
$$

$$
\begin{aligned}
&\Pr(\text{exactly 2 of the 5 } A\text{s}) = \binom{2}{0}\sum_{i<j}^{5} \Pr(A_i \cap A_j) - \binom{3}{1}\sum_{i<j<k}^{5} \Pr(A_i \cap A_j \cap A_k)\\
&+\binom{4}{2}\sum_{i<j<k<\ell}^{5} \Pr(A_i \cap A_j \cap A_k \cap A_\ell) - \binom{5}{3}\Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
&\Pr(\text{exactly 3 of the 5 } A\text{s}) = \binom{3}{0}\sum_{i<j<k}^{5} \Pr(A_i \cap A_j \cap A_k) - \binom{4}{1}\sum_{i<j<k<\ell}^{5} \Pr(A_i \cap A_j \cap A_k \cap A_\ell)\\
&+\binom{5}{2}\Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)
\end{aligned}
$$

$$
\Pr(\text{exactly 4 of the 5 } A\text{s}) = \binom{4}{0}\sum_{i<j<k<\ell}^{5} \Pr(A_i \cap A_j \cap A_k \cap A_\ell) - \binom{5}{1}\Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)
$$

and of course

$$\Pr(\text{exactly 4 of the 5 } A\text{s}) = \Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$$

Proof. As an example, we derive only the second one of these; finding the rest of them would follow a similar pattern.

Consider (1) with B_1, B_2 and B_3 replaced by A_3, A_4 and A_5, together with all its 'symmetric' counterparts (i.e. using A_1 and A_3 as the un-barred events, then A_1 and A_4, ..., for the total of $\binom{5}{2}$ equations). Obviously the corresponding events (i.e. $A_1 \cap A_2 \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}$, $A_1 \cap A_3 \cap \overline{A_2} \cap \overline{A_4} \cap \overline{A_5}, ...$) are all mutually exclusive, and their union yields the event of 'exactly 2 of the 5 As'. Adding the right hand sides of (1) yields the probability we need, resulting in $\binom{5}{2} \cdot \binom{3}{0}$ terms with 2 As, $\binom{5}{2} \cdot \binom{3}{1}$ terms with 3 As, $\binom{5}{2} \cdot \binom{3}{2}$ terms with 4 As, and $\binom{5}{2} \cdot \binom{3}{3}$ terms with 5 As. Since there are only $\binom{5}{2}$ *different* terms with 2 As, $\binom{5}{3}$ *different* terms with 3 As, $\binom{5}{4}$ *different* terms with 4 As, and $\binom{5}{5}$ *different* terms with 5 As, the amount of 'duplicity' for each of these is

$$\frac{\binom{5}{2} \cdot \binom{3}{0}}{\binom{5}{2}} = \binom{2}{0}, \quad \frac{\binom{5}{2} \cdot \binom{3}{1}}{\binom{5}{3}} = \binom{3}{1}, \quad \frac{\binom{5}{2} \cdot \binom{3}{2}}{\binom{5}{4}} = \binom{4}{2} \quad \text{and} \quad \frac{\binom{5}{2} \cdot \binom{3}{3}}{\binom{5}{5}} = \binom{5}{3}$$

respectively, and these are thus the correct coefficients of the resulting expansion (2).

In fully general terms, this argument boils down to

$$\frac{\binom{k}{m} \cdot \binom{k-m}{i}}{\binom{k}{m+i}} = \binom{m+i}{i}$$

■

Finally, these imply (by simple adding) that

$$\Pr(\text{at least 1 of the 5 } A\text{s}) = \sum_{i=1}^{5} \Pr(A_i) - \sum_{i<j}^{5} \Pr(A_i \cap A_j) + \sum_{i<j<k}^{5} \Pr(A_i \cap A_j \cap A_k)$$
$$- \sum_{i<j<k<\ell}^{5} \Pr(A_i \cap A_j \cap A_k \cap A_\ell) + \Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) \quad (3)$$

(we knew this already),

$$\Pr(\text{at least 2 of the 5 } A\text{s}) = \sum_{i<j}^{5} \Pr(A_i \cap A_j) - 2 \sum_{i<j<k}^{5} \Pr(A_i \cap A_j \cap A_k)$$
$$+3 \sum_{i<j<k<\ell}^{5} \Pr(A_i \cap A_j \cap A_k \cap A_\ell) - 4 \Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$$

$$\Pr(\text{at least 3 of the 5 } A\text{s}) = \binom{2}{0} \sum_{i<j<k}^{5} \Pr(A_i \cap A_j \cap A_k) - \binom{3}{1} \sum_{i<j<k<\ell}^{5} \Pr(A_i \cap A_j \cap A_k \cap A_\ell)$$
$$+\binom{4}{2} \Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$$

and

$$\Pr(\text{at least 4 of the 5 } A\text{s}) = \binom{3}{0} \sum_{i<j<k<\ell}^{5} \Pr(A_i \cap A_j \cap A_k \cap A_\ell) - \binom{4}{1} \Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$$

(at least 5 is of course the same as exactly 5).

Proof. In fully general terms, this boils down to showing that

$$\sum_{i=0}^{j} (-1)^{\ell-i} \binom{\ell}{i} = (-1)^{\ell-j} \binom{\ell-1}{j}$$

which is clearly true for $j = 0$. Assuming that it holds with j, to prove its correctness for $j \to j+1$ requires

$$(-1)^{\ell-j-1} \binom{\ell}{j+1} = (-1)^{\ell-j-1} \binom{\ell-1}{j+1} - (-1)^{\ell-j} \binom{\ell-1}{j}$$

or

$$\binom{\ell}{j+1} = \binom{\ell-1}{j+1} + \binom{\ell-1}{j}$$

which is a well known identity. ■