BROCK UNIVERSITY

Final Examination: December 2010	Number of Pages: 4
Course: MATH 2P81	Number of students: 60
Date of Examination: Dec. 9, 20010	Number of Hours: 3
Time of Examination: 9:00-12:00	Instructors: J Vrbik and N Pshenitsyna

Two sheet of notes, and the use of Maple, are permitted.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Translation dictionaries (e.g. English-French) or other dictionaries (thesaurus, definitions, technical) are not allowed unless specified by the instructor and indicated on the examination paper.

Full credit given for 7 complete answers.

Numerical answers must be correct to 4 significant digits.

1. Given that

$$P(A) = 0.50, \quad P(B) = 0.54, \quad P(C) = 0.53, \quad P(A \cap B) = 0.21,$$

$$P(A \cap C) = 0.25, \quad P(B \cap C) = 0.30 \text{ and } P(A \cap B \cap C) = 0.12$$

find

$$\Pr[(A \cup \bar{B} \cup C) \cap (\bar{A} \cup B \cup \bar{C})]$$

2. If A, B, C and D are mutually *independent*, and $\Pr(A) = 0.42$, $\Pr(B) = 0.23$, $\Pr(C) = 0.81$ and $\Pr(D) = 0.54$, find

$$\Pr[\bar{A} \cap (B \cup C) \cap (C \cup D)]$$

3. Four males and eight females are randomly seated at a round table. What is the probability that
   a. all males have only female neighbours (same as: each male has a female to his right),
   b. at least 5 females have no male neighbour (the females must either sit all together, or one be separated from the rest),
   c. no male sits between two females (the males must either sit together or in two pairs - be careful when the pairs are opposite each other).

4. Five cards are randomly dealt from the standard deck, a coin is flipped once if the cards are all singlets (i.e. all are of a distinct value - we are ignoring suits here), twice if there is at least one pair, three times for a triplet (including full house), and four times for a quadruplet of identical values (such as 4 jacks). Compute
   a. the probability of getting at least two heads ,
   b. the conditional probability of having dealt a triplet (in the first part of the experiment), given that we have obtained at least two heads.
5. Consider two integer-valued random variables X and Y having the following probability function
$$f_{xy}(i,j) = \frac{c}{1+i^2+j^2} \quad \text{when} \quad -2 \le i \le 1 \quad \text{and} \quad i \le j \le 1$$
   Find
   a. the value of c,
   b. $\mathrm{Var}(X)$ and $\mathrm{Cov}(X,Y)$,
   c. $\mathbb{E}\big(3X+2 \mid Y=0\big)$.
6. Customers arrive, randomly, at the average rate of 9.3 per hour. Let X be the number of customers arriving during the next 17 minutes, and Y be the time of arrival (in *minutes*) of the 6th arrival from now. Find
   a. the mean and standard deviation of X, and also of Y (these are two separate questions),
   b. $\Pr(X > 5)$ and, also, $\Pr(Y > 25 \text{ min.})$,
   c. the probability generating function of $2X-3$, and the moment generating function of $3-2Y$.

7. Consider randomly permuting the letters p r o b l e m a t i c. If X_T is the number of these letters which will be placed back in their original position, find
   a. the mean and standard deviation of X_T. Hint: $X_T = X_1 + X_2 + \ldots + X_{11}$, where X_1 equals to 1 or 0, depending on whether 'p' is placed back in its original location or not, similarly X_2 is based on 'r' falling in the second slot or not, etc. Then, find $\mathbb{E}(X_1)$, $\text{Var}(X_1)$ and $\text{Cov}(X_1, X_2)$, and put it all together.
   b. $\Pr(X_T \geq 2)$.
8. Consider a random variable X having the following probability generating function
$$P_x(z) = \frac{1}{\sqrt{3 - 2e^{z-1}}}$$
Find its mean, standard deviation, $\Pr(2 < X < 6)$, and the probability generating function of $3X + 2$. Also, compute (exactly - no approximation): $\Pr(2 < \bar{X} < 6)$ where $\bar{X}$ is the sample mean of 7 independent values drawn from this distribution. Hint: Express the same question in terms of $\sum_{i=1}^{7} X_i$.
9. Let X be a random variable with the following probability density function:
$$f(x) = \frac{c}{(x^2 - 2x + 3)^2}$$
where x is any real number. Find
   a. the value of c,
   b. the mean, standard deviation and the median of X.
   c. Suppose we draw a random independent sample of size 50 from this distribution. Using the Normal approximation, estimate $\Pr(0.9 < \bar{X} < 1.08)$, where $\bar{X}$ is the corresponding sample mean.

10. Consider paying \$20 to play the following game: There are 9 red, 12 blue and 8 green marbles in a box. Seven of them are randomly drawn *without* replacement. The player receives \$5 for each red marble and \$3 for each blue marble drawn. What is the probability of winning (net, total) at least \$10 in
    a. one round of this game,
    b. ten independent rounds of this game (use probability generating function),
    c. hundred independent rounds of this game (use Normal approximation).
11. Let X_1, X_2 and X_3 be three *independent* random variables, each having the exponential distribution with the mean of 2.3, 5.1 and 4.0, respectively. Compute
    a. the mean and variance of $Y \equiv 2X_1 - 3X_2 + X_3 - 7$,
    b. the covariance between Y (of part a) and $4X_1 + 2X_2 - X_3 + 3$,
    c. $\Pr[\min(X_1, X_2, X_3) < 4]$ - this means at least one of the three random variables must be less than 4,
    d. $\Pr[\max(X_1, X_2, X_3) > 6]$ - this means at least one of the three random variables must be bigger than 6.
    Hint: For both c. and d., the complement should be a lot easier.