BROCK UNIVERSITY

Final Examination: December 2012	Number of Pages: 4
Course: MATH 2P81	Number of students: 54
Date of Examination: Dec. 15, 2012	Number of Hours: 3
Time of Examination: 8:00-12:00	Instructor: J. Vrbik

Two sheet of notes and use of Maple are permitted.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Use or possession of unauthorized materials will automatically result in the award of a zero grade for this examination.

Full credit given for 14 (out of 22) complete answers.

Numerical answers must be correct to 4 significant digits.

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1. Given that

$$\begin{aligned} P(A) &= 0.50, \quad P(B) = 0.54, \quad P(C) = 0.53, \quad P(A \cap B) = 0.21, \\ P(A \cap C) &= 0.25, \quad P(B \cap C) = 0.30 \text{ and } P(A \cap B \cap C) = 0.12 \end{aligned}$$

   find

   (a)
$$\Pr\left[\overline{A \cap B} \cap \overline{A \cap C}\right]$$

   (b)
$$\Pr\left[\overline{A \cup B} \cap \overline{A \cup C}\right]$$

2. If A, B, C and D are mutually *independent*, and $\Pr(A) = 0.42$, $\Pr(B) = 0.23$, $\Pr(C) = 0.81$ and $\Pr(D) = 0.54$, find

   (a)
$$\Pr\left[(A \cup B \cup C \cup D\right]$$

   (b)
$$\Pr\left[\overline{A \cap B \cap C} \cap \overline{C \cap D}\right]$$

   Hint: In all 4 previous questions, computing $1 - \Pr(\text{of the complement})$ is easier.

3. A team wins (loses or ties) a game with the probability of 0.42 (0.38 or 0.20) respectively. It gets 3 points for each win and 1 point for each tie (no points for a loss). It plays a series of 12 independent games. Find

   (a) the expected value (and the corresponding standard deviation) of the total number of points it receives,

   (b) the probability of getting more than 20 points (use the corresponding PGF and Maple).

4. Let X_1 (X_2) be the number of dots shown in the first (second) roll of a die. A random hand of $\max(X_1, X_2)$ cards is then dealt from the usual deck of 52 cards.

   (a) Find the probability of dealing at least one spade.

   (b) Given that at least one spade has been dealt, what is the conditional probability that $\max(X_1, X_2)$ was greater than 3.

   Hint: Use a probability tree with 6 branches for the outcome of $\max(X_1, X_2)$ - i.e. the bigger of the two numbers shown - followed by 2 branches corresponding to dealing either 0 or at least 1 spade.

5. Let X_1, X_2 and X_3 be independent, exponentially distributed random variables, with expected values of 2.3, 4.1 and 3.3 hours respectively. Compute

   (a) $\text{Cov}(3X_1 - 2X_2 + X_3 - 7, X_1 + 4X_2 - 3X_3 + 3)$,

   (b) $\Pr[\min(X_1, X_2, X_3) < 52 \text{ min}]$.

6. The joint probability function of X and Y is given by

$$f_{xy}(i, j) = c \cdot (5 - i - j)$$

   where c is an appropriate constant (find it first), and i and j are both *positive* integers, such that $i + j < 4$. Compute:

   (a) the correlation coefficient between X and Y,

   (b) $\mathbb{E}\left(X^2 | Y = 2\right)$.

7. Customers arrive at the rate of 9.4 per hour. The current time is 8:07. Find

   (a) the expected time of arrival of the 6^{th} customer from now (give the answer in the hour:minute:second format) and the corresponding standard deviation (in minutes and seconds),

   (b) the probability that the 6^{th} customer arrives between 8:42 and 9:05.

8. Let X be a random variable having the following *distribution function*:

$$F(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{1 - \cos x}{2} & 0 < x \le \pi \\ 1 & x > \pi \end{cases}$$

   (a) Find the expected value, standard deviation, skewness and kurtosis of X.

   (b) Suppose we draw a random independent sample of size 60 from this distribution. Using the Normal approximation, find $\Pr(1.5 < \bar{X} < 1.6)$, where $\bar{X}$ is the corresponding sample mean.

9. Consider paying \$4 to play the following game: four dice are rolled and you receive \$3 for each pair (of identical numbers shown), \$10 for a triplet and \$200 for a quadruplet.

   (a) Find the expected value and standard deviation of your net win (in a single round). Hint: Build a table of the corresponding distribution first (make sure your probabilities add up to 1).

   (b) Using Normal approximation, estimate the probability of winning some money (net total) after $10,000$ rounds of this game.

10. Pay \$1 to play the following game: a random hand of 7 cards is dealt, and you get ¢25 for each spade and ¢75 for each ace (\$1 for the ace of spades, since it is both a spade and an ace).

    (a) Find the expected value and standard deviation of your *net* win (in one round).

    (b) Using the Normal approximation, estimate the probability of *losing* (*net total*) more than \$5.75 in 60 independent rounds of this game (don't forget the continuity correction).

11. This is a continuation of the previous question: Assume, again, that 60 independent rounds of this game are played. Compute the (exact - no approximation!) probability of

    (a) losing \$1 (and getting back nothing) in more than 5 (out of the 60) rounds,

    (b) losing (net total) more than \$5.75 at the end of the 60 rounds (use PGF and Maple).