1.

(a)

$$\Pr\left[\overline{A\cap B}\cap\overline{A\cap C}\right]=1-\Pr\left[(A\cap B)\cup(A\cap C)\right]=$$
$$1-\Pr(A\cap B)-\Pr(A\cap C)+\Pr(A\cap B\cap C)=1-0.21-0.25+0.12=66\%$$

(b)

$$\Pr\left[\overline{A\cup B}\cap\overline{A\cup C}\right]=1-\Pr\left[A\cup B\cup A\cup C\right]=1-\Pr\left[A\cup B\cup C\right]=$$
$$1-\Pr(A)-\Pr(B)-\Pr(C)+\Pr(A\cap B)+\Pr(A\cap C)+\Pr(B\cap C)-\Pr(A\cap B\cap C)=$$
$$1-0.50-0.54-0.53+0.21+0.25+0.30-0.12=7\%$$

2.

(a)

$$\Pr\left[(A\cup B\cup C\cup D\right]=1-\Pr(\bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D})=1-0.58\times0.77\times0.19\times0.46=96.10\%$$

(b)

$$\Pr\left[\overline{A\cap B\cap C}\cap\overline{C\cap D}\right]=1-\Pr\left[(A\cap B\cap C)\cup(C\cap D)\right]=$$
$$1-\Pr(A\cap B\cap C)-\Pr(C\cap D)+\Pr(A\cap B\cap C\cap D)=$$
$$1-0.42\times0.23\times0.81-0.81\times0.54+0.42\times0.23\times0.81\times0.54=52.66\%$$

3. Define

$$S=3W+T$$

then use the corresponding multinomial formulas for the mean and variance of W (number of wins) and T (number of losses), and for $\mathrm{Cov}(W,T)$.

(a)

$$\mathbb{E}\left(S\right)=3\mathbb{E}\left(W\right)+\mathbb{E}\left(T\right)=3\times12\times0.42+12\times0.20=17.52$$
$$\sigma_S=\sqrt{3^2\mathrm{Var}(W)+\mathrm{Var}(T)+2\cdot3\cdot\mathrm{Cov}(W,T)}=$$
$$\sqrt{12\times(3^2\times0.42\times0.58+0.20\times0.80-2\times3\times0.42\times0.20)}=4.710$$

(b) Expanding

$$(0.42z^3+0.20z+0.38)^{12}$$

and collecting coefficients of powers of z from 21 to 36, we get 26.39%.

4. We know from class (and it's easy to derive, if necessary) that the distribution of $\max(X_1,X_2)$ is given by

$\max(X_1,X_2)$	1	2	3	4	5	6
Pr	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

or, equivalently, by

$$f(i) = \frac{2i-1}{36} \qquad i = 1, 2, ...6$$

The sample space of the experiment can then be represented by

1, 0s	$\frac{1}{36} \times \frac{\binom{39}{1}}{\binom{52}{1}}$	$\frac{1}{48}$
1, >1s	$\frac{1}{36} \times \left(1 - \frac{\binom{39}{1}}{\binom{52}{1}}\right)$	$\frac{1}{144}$ $\lhd$
2, 0s	$\frac{3}{36} \times \frac{\binom{39}{2}}{\binom{52}{2}}$	$\frac{19}{408}$
2, >1s	$\frac{3}{36} \times \left(1 - \frac{\binom{39}{2}}{\binom{52}{2}}\right)$	$\frac{5}{136}$ $\lhd$
3, 0s	$\frac{5}{36} \times \frac{\binom{39}{3}}{\binom{52}{3}}$	$\frac{703}{12240}$
3, >1s	$\frac{5}{36} \times \left(1 - \frac{\binom{39}{3}}{\binom{52}{3}}\right)$	$\frac{997}{12240}$ $\lhd$
4, 0s	$\frac{7}{36} \times \frac{\binom{39}{4}}{\binom{52}{4}}$	$\frac{703}{11900}$
4, >1s	$\frac{7}{36} \times \left(1 - \frac{\binom{39}{4}}{\binom{52}{4}}\right)$	$\frac{7249}{53550}$ $\lhd$ $\surd$
5, 0s	$\frac{9}{36} \times \frac{\binom{39}{5}}{\binom{52}{5}}$	$\frac{2109}{38080}$
5, >1s	$\frac{9}{36} \times \left(1 - \frac{\binom{39}{5}}{\binom{52}{5}}\right)$	$\frac{7411}{38080}$ $\lhd$ $\surd$
6, 0s	$\frac{11}{36} \times \frac{\binom{39}{6}}{\binom{52}{6}}$	$\frac{7733}{157920}$
6, >1s	$\frac{11}{36} \times \left(1 - \frac{\binom{39}{6}}{\binom{52}{6}}\right)$	$\frac{121561}{473760}$ $\lhd$ $\surd$

(a)

$$\frac{1}{144} + \frac{5}{136} + \frac{997}{12240} + \frac{7249}{53550} + \frac{7411}{38080} + \frac{121561}{473760} = \frac{3371\,923}{4737\,600} = 71.17\%$$

Or, more elegantly, in one line (using Maple):

$$1 - \sum_{i=1}^{6} \frac{2i-1}{36} \cdot \frac{\binom{39}{i}}{\binom{52}{i}} = 71.17\%$$

(b)

$$\frac{\frac{7249}{53550} + \frac{7411}{38080} + \frac{121561}{473760}}{\frac{3371\,923}{4737\,600}} = 82.41\%$$

or

$$\frac{\sum_{i=4}^{6} \frac{2i-1}{36} \cdot \left(1 - \frac{\binom{39}{i}}{\binom{52}{i}}\right)}{\sum_{i=1}^{6} \frac{2i-1}{36} \cdot \left(1 - \frac{\binom{39}{i}}{\binom{52}{i}}\right)} = 82.41\%$$

5.

(a) Since the covariances are equal to 0 (independence), we get

$$\text{Cov}(3X_1 - 2X_2 + X_3 - 7, X_1 + 4X_2 - 3X_3 + 3) = 3\text{Var}(X_1) - 8\text{Var}(X_2) - 3\text{Var}(X_3) = 3 \times 2.3^2 - 8 \times 4.1^2 - 3 \times 3.3^2 = -151.28$$

(b)

$$\begin{aligned} 1 - \Pr(X_1 &> 52 \text{ min}) \cdot \Pr(X_2 > 52 \text{ min}) \cdot \Pr(X_3 > 52 \text{ min}) \\ &= 1 - \exp\left(-\frac{52}{60} \times (\frac{1}{2.3} + \frac{1}{4.1} + \frac{1}{3.3})\right) = 57.29\% \end{aligned}$$

6.

$Y \downarrow$ $X \rightarrow$	1	2	
1	$3c$	$2c$	$\frac{5}{7}$
2	$2c$	0	$\frac{2}{7}$
	$\frac{5}{7}$	$\frac{2}{7}$	

where $c = \frac{1}{7}$.

(a)

$$\begin{aligned} \mu_x &= \mu_y = 1 \times \frac{5}{7} + 2 \times \frac{2}{7} = \frac{9}{7} \\ Var(X) &= Var(Y) = 1^2 \times \frac{5}{7} + 2^2 \times \frac{2}{7} - (\frac{9}{7})^2 = \frac{10}{49} \\ Cov(X,Y) &= 1 \times \frac{3}{7} + 2 \times \frac{2}{7} + 2 \times \frac{2}{7} - (\frac{9}{7})^2 = -\frac{4}{49} \\ \rho &= -\frac{4}{49} \div \frac{10}{49} = -\frac{1}{5} \end{aligned}$$

(b)

$X \mid Y = 2$	1
Pr	1

$$\mathbb{E}\left(X^2 | Y = 2\right) = 1$$

7.

(a)

$$\begin{aligned} \mu &= 8 + \frac{7}{60} + \frac{6}{9.4} = 8.755 \text{ hours} \\ \sigma &= \frac{\sqrt{6}}{9.4} = 0.26058 \text{ hours} \end{aligned}$$

which translate to 8:45:18, and 15 min and 38.088 sec.

(b) Using the corresponding gamma PDF:

$$\frac{9.4^6}{5!}\int_{\frac{35}{60}}^{\frac{58}{60}} x^5 \exp(-9.4\cdot x)dx = 42.125\%$$

or (more clumsily), via Poisson

$$\begin{aligned}
\Lambda_1 &= \frac{35}{60}\times 9.4\\
\Lambda_2 &= \frac{58}{60}\times 9.4\\
e^{-\Lambda_1}\sum_{i=0}^{5}\frac{\Lambda_1^i}{i!} &- e^{-\Lambda_2}\sum_{i=0}^{5}\frac{\Lambda_2^i}{i!} = 42.125\%
\end{aligned}$$

8. First convert $F(x)$ to the corresponding PDF:

$$f(x) = \frac{\sin x}{2} \qquad 0 < x < \pi$$

(a) Then, routinely

$$\begin{aligned}
\mu &= \int_0^{\pi} x\cdot\frac{\sin x}{2}dx = \frac{\pi}{2}\\
\sigma^2 &= \int_0^{\pi}(x-\frac{\pi}{2})^2\frac{\sin x}{2}dx = \frac{1}{4}\pi^2 - 2 = 0.4674\\
&\int_0^{\pi}(x-\frac{\pi}{2})^3\frac{\sin x}{2}dx/(\frac{1}{4}\pi^2-2)^{3/2} = 0\\
&\int_0^{\pi}(x-\frac{\pi}{2})^4\frac{\sin x}{2}dx/(\frac{1}{4}\pi^2-2)^{2} = 2.194
\end{aligned}$$

(b) For $\bar{X}$, the mean stays the same and the variance becomes 60 times smaller.

$$\int_{1.5}^{1.6}\exp\left(-\frac{(x-\pi/2)^2}{2(\frac{1}{4}\pi^2-2)}\cdot 60\right)dx\Big/\sqrt{2\pi\cdot\frac{\frac{1}{4}\pi^2-2}{60}} = 41.84\%$$

9. Visualize the sample space of 6^4 equally likely outcomes; P_4^6 are all different, $6\cdot C_2^5\cdot\binom{4}{2,1,1}$ have one pair, $C_2^6\cdot\binom{4}{2,2}$ have two, $6\cdot 5\cdot\binom{4}{3,1}$ have one triplet, and 6 of them are a quad. This implies:

(a)

W	-4	-1	2	6	196
Pr	$\frac{5}{18}$	$\frac{5}{9}$	$\frac{5}{72}$	$\frac{5}{54}$	$\frac{1}{216}$

$$\begin{aligned}
\mathbb{E}(W) &= -4\times\frac{5}{18} - 1\times\frac{5}{9} + 2\times\frac{5}{72} + 6\times\frac{5}{54} + 196\times\frac{1}{216} = -\frac{7}{108} \text{ dollars}\\
\sigma &= \sqrt{16\cdot\frac{5}{18}+\frac{5}{9}+4\cdot\frac{5}{72}+36\cdot\frac{5}{54}+196^2\cdot\frac{1}{216}-\frac{(-7)^2}{108^2}} = 13.655 \text{ dollars}
\end{aligned}$$

(b) The sum of independent 10,000 outcomes has both the mean and the variance 10,000 bigger. Thus:

$$\frac{1}{\sqrt{20000\pi}\cdot 13.655}\int_{0.5}^{\infty}\exp\left(-\frac{(s+\frac{70000}{108})^2}{20000\cdot 13.655^2}\right)ds = 31.74\%$$

10. Introduce

$$W = \frac{1}{4}U + \frac{3}{4}V - 1$$

where U is the number of spades and V is the number of aces dealt (note that there is no need to consider the ace of spades separately; the formula takes care of it automatically). Now using the hypergeometric formulas for the mean and variance of each U and V, and for the covarinace of U and V (here, one must use the 'overlap' formula, getting 0), one gets:

(a)

$$\mathbb{E}(W) = \frac{1}{4}\mathbb{E}(U) + \frac{3}{4}\mathbb{E}(V) - 1 =$$
$$7\cdot\left(\frac{1}{4}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{13}\right) - 1 = -\frac{33}{208} = -¢15.865$$
$$\sigma_w = \sqrt{\frac{1}{4^2}\mathrm{Var}(U) + (\frac{3}{4})^2\mathrm{Var}(V)} =$$
$$\sqrt{7\cdot\left(\frac{1}{4^2}\cdot\frac{1}{4}\cdot\frac{3}{4}+(\frac{3}{4})^2\cdot\frac{1}{13}\cdot\frac{12}{13}\right)\cdot\frac{45}{51}} = \sqrt{\frac{234\,675}{735\,488}} = ¢56.49$$

(b) Again (as in Q. 9), the total has a mean and variance which are both 60 times bigger. The continuity correction means using the value half way between $-\$6$ and $-\$5.75$, namely:

$$\int_{-\infty}^{-5.875}\exp\left(-\frac{(x+\frac{33}{208}\cdot 60)^2}{2\cdot\frac{234\,675}{735\,488}\cdot 60}\right)dx\Big/\sqrt{2\pi\cdot\frac{234\,675}{735\,488}\cdot 60} = 79.75\%$$

11.

(a) The corresponding distribution (of the number of 'successes') is binomial with $n = 60$ and p being the probability of getting no spades and no aces, i.e. $p = \frac{\binom{36}{7}}{\binom{52}{7}}$. This yields:

$$\sum_{i=6}^{60}\binom{60}{i}\times\left(\frac{\binom{36}{7}}{\binom{52}{7}}\right)^i\times\left(1.0-\frac{\binom{36}{7}}{\binom{52}{7}}\right)^{60-i} = 17.02\%$$

(b) This time, we have to take \$0.25 as our monetary unit. Then, by expanding

$$\left(\sum_{j=0}^{1}\sum_{k=0}^{3}\sum_{i=0}^{7-j-k} z^{i+4j+3k-4}\,\frac{\binom{12}{i}\binom{1}{j}\binom{3}{k}\binom{36}{7-i-j-k}}{\binom{52.0}{7}}\right)^{60}$$

and adding coefficients of all powers of z *smaller* than -23 (this corresponds to $-\$5.75$), we get 79.87%.