

BROCK UNIVERSITY

Final Examination: December 2006
 Course: MATH 2P81
 Date of Examination: Dec. 7, 2006
 Time of Examination: 19:00 -22:00

Number of Pages: 4
 Number of students: 36
 Number of Hours: 3
 Instructor: J. Vrbik

Two sheet of notes, and the use of Maple, are permitted.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Translation dictionaries (e.g. English-French) or other dictionaries (thesaurus, definitions, technical) are not allowed unless specified by the instructor and indicated on the examination paper.

Full credit given for 8 complete answers.

Numerical answers must be correct to 4 significant digits.

1. Given that

$$\begin{aligned} P(A) &= 0.31, & P(B) &= 0.33, & P(C) &= 0.37 \\ P(A \cap B) &= 0.11, & P(A \cap C) &= 0.13, & P(B \cap C) &= 0.12 \\ && \text{and } P(A \cap B \cap C) &= 0.05 \end{aligned}$$

find

$$\Pr [(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)]$$

2. If A , B , C and D are mutually *independent*, and $\Pr(A) = 0.47$, $\Pr(B) = 0.21$, $\Pr(C) = 0.83$ and $\Pr(D) = 0.55$, find

$$\Pr [(A \cap \bar{B} \cap \bar{C} \cap D) \cup (\bar{A} \cap B \cap C) \cup (\bar{A} \cap B \cap D)]$$

3. Three blue, two red and five green beads are randomly arranged in a circular necklace. What is the probability that the blue beads all end up next to each other (forming a group of three consecutive blue beads), while the two red beads are separated by at least one bead of another colour.

4. Let X and Y be two integer-valued random variables with the following joint probability function:

$$f_{XY}(i, j) = c \cdot j^i \quad \text{where} \quad -2 \leq i \leq 2 \quad \text{while} \quad 1 \leq j \leq 1 + |i|$$

Find:

- (a) the value of c ,
 - (b) $\mathbb{E}(X)$ and $\mathbb{E}(Y)$,
 - (c) $\text{Cov}(X, Y)$,
 - (d) $\mathbb{E}(Y \mid X = -2)$.
5. Let X be a random variable with the following probability density function:

$$f(x) = \begin{cases} c \cdot x^2 & 0 < x \leq 2 \\ c \cdot (3 - x) & 2 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Find:

- (a) the value of c ,
 - (b) the distribution function $F(x)$,
 - (c) $\Pr(1 < X < 2.5)$,
 - (d) the mean and variance of this distribution.
6. Let X and Y have a bi-variate distribution described by the following joint probability density function:

$$f(x, y) = c \cdot x \quad \text{for} \quad x > 0, y > 0 \text{ and } x^2 + y^2 < 1,$$

zero otherwise. Find:

- (a) the value of c ,
- (b) $\Pr(X + Y > 0.5)$,
- (c) $\mathbb{E}\left(\frac{Y^2}{X}\right)$,
- (d) $\mathbb{E}(X \mid Y = 0.5)$.

7. Consider paying \$21 to play the following game: 8 cards are randomly dealt from an ordinary deck, and you receive \$20 for each ace, \$10 for each king, and \$1 for each spade in this hand (for the king of spades, you thus collect \$11, for the ace of spades \$21). Calculate your expected net win and the corresponding standard deviation. What is the probability of breaking even?
8. There are two dice, one regular, the other 'loaded' in the following manner:

# of dots:	1	2	3	4	5	6
Pr:	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$

One of the two dice is selected at random and rolled 12 times.

- (a) What is the probability of getting more than 2 sixes?
 - (b) What is the probability that each number of dots appears exactly twice?
 - (c) Given that we have obtained the last outcome (each number appearing exactly twice), what is the conditional probability of having selected the regular die?
9. Let X and Y be two *independent* random variables, both having the exponential distribution, but with different means: X has a mean of 3, Y has a mean of 5.
- (a) Compute $\Pr(X > Y)$.
 - (b) What is the expected value and standard deviation of $Y - 3X + 2$?
 - (c) Find $\text{Cov}(Y - 3X + 2, 5X - 2Y + 1)$.

10. The Earth is hit by a 'big' asteroid (more than 100 meters in diameter) on the average once every 27 thousand years. Find the probability that
- (a) the next such event will happen less than 10 thousand years from now,
 - (b) the third such event will happen between 50 and 80 thousand years from now,
 - (c) more than 5 big asteroids will hit the Earth during the next 100 thousand years.

11. A random variable X has the following moment generating function:

$$M(t) = \frac{e^t}{(1 - 2t)^3}$$

Find

- (a) the mean and variance of X ,
- (b) the moment generating function of $Y \equiv 3 - 2X$,
- (c) the mean and variance of \bar{X} (the **sample mean** of a random independent sample of size 20),
- (d) the moment generating function of \bar{X} .