MATH2P81 SECOND MIDTERM November 15, 2001
Full credit given for three correct and complete answers.
Final answers must be correct to at least 4 significant digits.
Open-book exam. Duration: 50 minutes

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1. Consider flipping a coin three times. Let X be the total number heads, let Y be the number of times the outcome changes (from H to T, or from T to H; e.g. TTT will yield a value of 0, THT the value of 2, etc.). Construct a table of the joint distribution of X and Y.

2. Based on the previous table, find

   (a) $\mathbb{E}\left(\frac{1}{1+X} \,|\, Y=1\right)$

   (b) $\mathrm{Cov}(X,Y)$

3. Flipping four coins and getting the same number of heads as tails (2 each) is considered a success. This experiment (trial) is repeated till 3 successes are obtained. Identify (i.e. give the name of) the distribution of the total number of trials, and specify the value of its two parameters. Then compute:

   (a) The expected number of trials needed, and the corresponding standard deviation.

   (b) Probability that this will take exactly 5 trials.

   (c) At least 15 trials..

4. Five cards are dealt from a standard deck of 52. Let X (Y) be the number of spaces (diamonds) obtained, respectively. Identify the joint distribution of X and Y. Then find:

   (a) $\Pr(X+Y=3)$
   Hint: Consider spades **and** diamonds to be 'special'.

   (b) $\Pr(X=Y)$.

   (c) $\mathrm{Var}(3X-2Y-4)$.

5. Consider rolling a die 9 times. Let X (Y) be the number of sixes (odd values) obtained, respectively. Identify the joint distribution of X and Y. Then compute:

   (a) $\Pr(X > 2)$.

   (b) $\Pr(X = 2 \cap Y = 4)$.

   (c) $\mathrm{Cov}(X + Y - 1, X - Y + 1)$.

Solution sheet

1.

X ▶ / Y ▼	0	1	2	3	
0	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\frac{2}{8}$
1	0	$\frac{2}{8}$	$\frac{2}{8}$	0	$\frac{4}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{2}{8}$
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

2.

$X \mid Y=1$	1	2
Pr:	$\frac{1}{2}$	$\frac{1}{2}$

(a) $\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}=\frac{5}{12}=0.4167$

(b) $\mathbb{E}(X\cdot Y)-\mu_X\cdot\mu_Y=\frac{12}{8}-\frac{3}{2}\cdot 1=0$

3. Negative binomial, with $k=3$ and $p=\binom{4}{2}\cdot(\frac{1}{2})^2\cdot(\frac{1}{2})^2=\frac{3}{8}$

(a) $\mu=\frac{3}{3/8}=8$, $\sigma=\sqrt{8\cdot(\frac{8}{3}-1)}=\sqrt{\frac{40}{3}}=3.651$

(b) $\binom{4}{2}\cdot(\frac{3}{8})^3\cdot(\frac{5}{8})^2=12.36\%$

(c) $(\frac{5}{8})^{14}+14\cdot\frac{3}{8}\cdot(\frac{5}{8})^{13}+\binom{14}{2}\cdot(\frac{3}{8})^2\cdot(\frac{5}{8})^{12}=5.851\%$

4. Multivariate hypergeometric

(a) $\dfrac{\binom{26}{3}\cdot\binom{26}{2}}{\binom{52}{5}}=32.51\%$

(b) $\dfrac{\binom{13}{0}\cdot\binom{13}{0}\cdot\binom{26}{5}+\binom{13}{1}\cdot\binom{13}{1}\cdot\binom{26}{3}+\binom{13}{2}\cdot\binom{13}{2}\cdot\binom{26}{1}}{\binom{52}{5}}=25.52\%$

(c) $9\cdot\text{Var}(X)+4\cdot\text{Var}(Y)-12\cdot\text{Cov}(X,Y)=5\cdot\big[9\cdot\frac{1}{4}\cdot\frac{3}{4}+4\cdot\frac{1}{4}\cdot\frac{3}{4}+12\cdot\frac{1}{4}\cdot\frac{1}{4}\big]\cdot\frac{47}{51}=14.69$

5. Multinomial

(a) $1-\big[(\frac{5}{6})^9+9\cdot\frac{1}{6}\cdot(\frac{5}{6})^8+\binom{9}{2}(\frac{1}{6})^2(\frac{5}{6})^7\big]=17.83\%$

(b) $\binom{9}{2,4,3}(\frac{1}{6})^2(\frac{1}{2})^4(\frac{1}{3})^3=8.102\%$

(c) $\text{Var}(X)-\text{Var}(Y)=9\cdot\frac{1}{6}\cdot\frac{5}{6}-9\cdot\frac{1}{2}\cdot\frac{1}{2}=-1$