1 Basic algebra

A few formulas:

$$
\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
&\vdots
\end{aligned}
$$

The coefficients follow from PASCAL'S TRIANGLE, the expansion is called BINOMIAL.

Also:

$$
\begin{aligned}
a^2 - b^2 &= (a-b)\,(a+b) \\
a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\
&\vdots
\end{aligned}
$$

(do you know how to continue)?

Algebraic rules: Both *addition* and *multiplication*, individually, are COMMUTATIVE and ASSOCIATIVE (the important implication is that we don't need parentheses when multiplying 3 or more terms).

Considered together, they follow the DISTRIBUTIVE law, namely

$$
\begin{aligned}
&(a+b+c)(d+e+f+h) = \\
&ad + ae + af + ah + bd + be + bf + bh + cd + ce + cf + ch
\end{aligned}
$$

(each term from the first set of parentheses multiplies each term of the second set, for the total of 3×4 terms) What if we have 3 or more sets of parentheses to multiply?.

Note that division and exponentiation are neither commutative nor associative.

Polynomial: A sum of several powers of a variable (say z) multiplied by a COEFFICIENT, e.g. $3 - 2z + 4z^2 - 5z^3$. The highest power is the polynomial's DEGREE. We should know how to add, multiply and raise polynomials to an integer power (how about division?). We often need to deal with large-size polynomials - that's why we use Maple.

Exponentiation and related rules:

$$
\begin{aligned}
a^A \cdot a^B &= a^{A+B} \\
(a^A)^B &= a^{AB}
\end{aligned}
$$

Also note that

$$(a^A)^B \neq a^{(A^B)}$$

(not associative).

Logarithm (base a) is the solution to

$$a^x = A$$

denoted

$$x = \log_a A$$

(i.e. it is the INVERSE FUNCTION to exponentiation). When $a = e$ $(= 2.7183...)$, this is written as

$$x = \ln A$$

and called NATURAL LOGARITHM. Its basic rules are:

$$\begin{aligned} \ln(A \cdot B) &= \ln A + \ln B \\ \ln(A^B) &= B \cdot \ln A \end{aligned}$$

2 Geometric (and other) series

The *finite* version is

$$1 + a + a^2 + a^3 + ... + a^N = \frac{1 - a^{N+1}}{1 - a}$$

valid for all $a \neq 1$, (we don't need $a = 1$, correct?) and positive integer N. Would you remember the proof?

Can be extended to *infinite* series, but only when $|a| < 1$ (the issue of CONVERGENCE):

$$1 + a + a^2 + a^3 + a^4 + ... = \frac{1}{1 - a}$$

There is one more infinite series (we will call it, informally, the EXPONENTIAL SERIES) worth knowing:

$$1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \frac{a^4}{4!} + ... = e^a$$

(this is how one 'discovers' e). The denominators are called FACTORIALS.

3 Some calculus

Differentiation of a FUNCTION of (say) x. Note on terminology: We *differentiate* (never say 'derive') a function, but the answer is called the function's DERIVATIVE.

The main formulas are

$$\begin{aligned} \frac{d}{dx}(x - a)^\beta &= \beta \cdot (x - a)^{\beta - 1} \\ \frac{d}{dx}(e^{\beta(x-a)}) &= \beta e^{\beta(x-a)} \end{aligned}$$

where a and β are two constants (β is not necessarily an integer). An alternate notation is $(e^{\beta x})' = \beta e^{\beta x}$, but here it must be *understood* what the independent variable is

There are three basic rules of differentiation (the PRODUCT, QUOTIENT and CHAIN rule), symbolically:

$$\begin{aligned}(f(x)\cdot g(x))' &= f'(x)\cdot g(x)+f(x)\cdot g'(x)\\ \left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g(x)^2}\\ f\left(g(x)\right)' &= f'\left(g(x)\right)\cdot g(x)\end{aligned}$$

where $f'(g(x))$ implies that f is differentiated with respect to its whole argument, i.e. $g(x)$.

The product rule can be extended to the second derivative and beyond:

$$(f(x)\cdot g(x))'' = f''(x)\cdot g(x)+2f'(x)\cdot g'(x)+f(x)\cdot g''(x)$$

$$(f(x)\cdot g(x))''' = f'''(x)\cdot g(x)+3f''(x)\cdot g'(x)+3f'(x)\cdot g''(x)+f(x)\cdot g'''(x)$$

etc. (Pascal's triangle again).

The important point is that any function can be differentiated any number of times (we use Maple).

Taylor (Maclaurin) Expansion of a function:

$$f(x)=f(0)+x\cdot f'(0)+\frac{x^2}{2}\cdot f''(0)+\frac{x^3}{3!}\cdot f'''(0)+\frac{x^4}{4!}\cdot f^{(4)}(0)+....$$

This is how one would prove our 'exponential series'. And two more examples:

$$\ln(1+x)=x-x^2/2+x^3/3-x^4/4+...$$

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+...$$

(this is an alternate proof of the infinite-geometric-series formula).

Basic limits (when $n\to\infty$)

One should know how to deal with a rational expression (go by the highest degree of numerator and denominator):

$$\lim_{n\to\infty}\frac{2n^2+3}{n^2-4n+1}=2$$

We will also need one rather special and important limit, namely:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

and its generalization

$$\lim_{n\to\infty}\left(1+\frac{a}{n}+\frac{b}{n^2}+...\right)^n=e^a\equiv\exp(a)$$

(introducing an alternate, Maple-like notation for e^a). Note that the b coefficient (and higher) does not affect the answer.

L'Hôpital rule (to deal with the $\frac{0}{0}$ case):

$$\lim_{x\to0}\frac{e^x-1}{x}=\lim_{x\to0}\frac{(e^x-1)'}{(x)'}=\lim_{x\to0}\frac{e^x}{1}=\frac{e^0}{1}=1$$

Integration

Basic formulas for indefinite (i.e. finding 'antiderivative') integration

$$\begin{aligned}\int (x+a)^\beta dx &= \frac{(x+a)^{\beta+1}}{\beta+1} \qquad \beta \neq -1 \\ \int \frac{dx}{x+a} &= \ln|x+a| \\ \int e^{\beta(x-a)} dx &= \frac{e^{\beta(x-a)}}{\beta}\end{aligned}$$

and techniques, such as

Change of variable:

$$\int (x^2+3)^5 \cdot 2x \, dx = \int (y+3)^5 dy = \frac{(y+3)^6}{6} = \frac{(x^2+3)^6}{6}$$

By-part integration:

$$\begin{aligned}\int \ln(x)\, dx &= \int 1 \cdot \ln(x)\, dx = \int (x)' \cdot \ln(x)\, dx = x \cdot \ln(x) - \int x \cdot (\ln(x))' \, dx \\ &= x \cdot \ln(x) - \int x \cdot \frac{1}{x}\, dx = x \cdot \ln(x) - \int 1\, dx = x \cdot \ln(x) - x\end{aligned}$$

Know how to convert these to a specific definite integral, e.g.

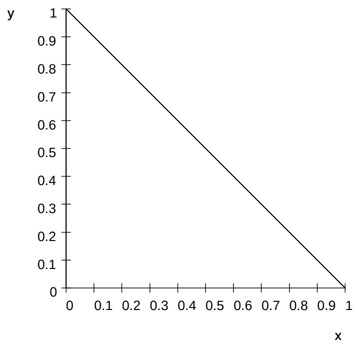
$$\begin{aligned}\int_0^1 \ln(x)\, dx = x \cdot \ln(x) - x\Big|_{x=0}^1 &= -1 - \lim_{x\to 0} x \cdot \ln(x) \\ = -1 - \lim_{x\to 0} \frac{(\ln(x))'}{\left(\frac{1}{x}\right)'} &= -1 - \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = -1 + \lim_{x\to 0} x = -1\end{aligned}$$

(represents area *above* the x axis).

Not all functions are 'analytically' integrable (we cannot express the result as a combination of the usual functions), but most definite integrals can be evaluated *numerically* (remember Simpson's rule?).

4 Two-dimensional geometry

We need to understand just one concept - how do we describe, *mathematically*, a two-dimensional region, such as the following triangle:



We can chose one of the following *equivalent* ways of doing it:

1. $0 \le y \le 1 - x$ (the CONDITIONAL range of the y values), where $0 \le x \le 1$ (followed by the MARGINAL range of possible values of x) $-$ visualize this as the triangle being filled with *vertical segments*, the marginal range being the projection of the triangle into the x axis.

2. $0 \le x \le 1 - y$ (conditional x-range), where $0 \le y \le 1$ (marginal y-range); we are now 'filling' the triangle with *horizontal* segments.

The fact that we can get the second description from the first just by the $x \leftrightarrow y$ interchange is only a coincidence (the region is a mirror image of itself with respect to the 45^{o} straight line).

Now, visualize the region described by $0 \le y \le 1 - x^2$ (vertical segments), where $0 \le x \le 1$. How would the 'horizontal' description look like? Answer: $0 \le x \le \sqrt{1 - y^2}$, where $0 \le y \le 1$.