Chebyshev Polynomials:

In our last example, the error increases towards each end. We can fix it by employing the following weight function:

$$W(X) = \frac{1}{\sqrt{1-X^2}}$$

(to make the procedure work harder at each end).

Now we want to minimize

$$S \equiv \int_{-1}^{1} \frac{\left(Y(X) - \hat{a}\Phi_0(X) - \hat{b}\Phi_1(X) - \hat{c}\,\Phi_2(X)\right)^2}{\sqrt{1-X^2}} dX$$

where $\Phi_0(X)$, $\Phi_1(X)$ and $\Phi_2(X)$ are polynomials of degree zero, one and two, orthogonal in a new way:

$$\int_{-1}^{1} \frac{\Phi_i(X)\Phi_j(X)\,dX}{\sqrt{1-X^2}} = 0$$

whenever $i \neq j$.

Then, the normal equations are

$$\left[\begin{array}{ccc|c} \int_{-1}^{1} \frac{\Phi_0(X)^2 dX}{\sqrt{1-X^2}} & 0 & 0 & \int_{-1}^{1} \frac{\Phi_0(X)Y(X) dX}{\sqrt{1-X^2}} \\ 0 & \int_{-1}^{1} \frac{\Phi_1(X)^2 dX}{\sqrt{1-X^2}} & 0 & \int_{-1}^{1} \frac{\Phi_1(X)Y(X)\,dX}{\sqrt{1-X^2}} \\ 0 & 0 & \int_{-1}^{1} \frac{\Phi_2(X)^2 dX}{\sqrt{1-X^2}} & \int_{-1}^{1} \frac{\Phi_2(X)Y(X)\,dX}{\sqrt{1-X^2}} \end{array}\right]$$

yielding the solution:

$$\begin{aligned} \hat{a} &= \frac{\int_{-1}^{1} \frac{\Phi_0(X)Y(X)dX}{\sqrt{1-X^2}}}{\alpha_0} \\ \hat{b} &= \frac{\int_{-1}^{1} \frac{\Phi_1(X)Y(X)dX}{\sqrt{1-X^2}}}{\alpha_1} \\ \hat{c} &= \frac{\int_{-1}^{1} \frac{\Phi_2(X)Y(X)dX}{\sqrt{1-X^2}}}{\alpha_2} \end{aligned}$$

where $\alpha_0 \equiv \int_{-1}^{1} \frac{\Phi_0(X)^2 dX}{\sqrt{1-X^2}}$, $\alpha_1 \equiv \int_{-1}^{1} \frac{\Phi_1(X)^2 dX}{\sqrt{1-X^2}}$,...

Typical-error:

$$\sqrt{\frac{\int_{-1}^{1} \frac{Y(X)^2 dX}{\sqrt{1-X^2}} - \alpha_0\,\hat{a}^2 - \alpha_1\,\hat{b}^2 - \alpha_2\,\hat{c}^2}{\pi}}$$

denominator is the total weight, i.e. $\int_{-1}^{1} \frac{dX}{\sqrt{1-X^2}}$.

Using Gram-Schmidt, we now construct Φ_0, Φ_1, ... Since weight function is symmetric, i.e. $W(X) = W(-X)$, we use the odd-even trick.

We also need

$$\int_{-1}^{1} \frac{X^n dX}{\sqrt{1-X^2}} = \frac{(2n-1)!!}{(2n)!!}\pi$$

when n is even (equal to zero for n odd).

The first polynomial is always

$$\Phi_0(X) = 1$$

with $\alpha_0 = \int_{-1}^{1} \frac{dX}{\sqrt{1-X^2}} = \pi$.

Similarly, the next one is

$$\Phi_1(X) = X$$

with $\alpha_1 = \int_{-1}^{1} \frac{X^2 dX}{\sqrt{1-X^2}} = \frac{\pi}{2}$.

$\Phi_2(X) = X^2 + C$, with C such that

$$\int_{-1}^{1} \frac{(X^2 + C)\, dX}{\sqrt{1-X^2}} = \pi C + \frac{\pi}{2} = 0 \Rightarrow C = -\frac{1}{2}$$

resulting in

$$\Phi_2(X) = X^2 - \frac{1}{2}$$

and $\alpha_2 = \int_{-1}^{1} \frac{(X^2-\frac{1}{2})^2 dX}{\sqrt{1-X^2}} = \frac{\pi}{8}$.

Similarly

$$\int_{-1}^{1} \frac{(X^3 + CX) \cdot X\, dX}{\sqrt{1-X^2}} = \frac{3\pi}{8} + \frac{\pi}{2}C = 0 \Rightarrow C = -\frac{3}{4}$$

which yields

$$\Phi_3(X) = X^3 - \frac{3}{4}X$$

and $\alpha_3 = \int_{-1}^{1} \frac{(X^3-\frac{3}{4}X)^2 dX}{\sqrt{1-X^2}} = \frac{\pi}{32}$.

Now, $\Phi_2(X) = X^4 + C_2\Phi_2(X) + C_0\Phi_0(X)$, where

$$\int_{-1}^{1} \frac{\left(X^4 + C_2\Phi_2(X) + C_0\Phi_0(X)\right) \cdot \Phi_2(X)\, dX}{\sqrt{1-X^2}}$$

$$= \frac{\pi}{8} + \frac{\pi}{8} C_2 = 0 \Rightarrow C_2 = -1$$

$$\int_{-1}^{1} \frac{\left(X^4 + C_2\Phi_2(X) + C_0\Phi_0(X)\right) \cdot \Phi_0(X)\, dX}{\sqrt{1-X^2}}$$

$$= \frac{3\pi}{8} + \pi C_0 = 0 \Rightarrow C_0 = -\frac{3}{8}$$

resulting in

$$\Phi_4(X) = X^4 - (X^2 - \frac{1}{2}) - \frac{3}{8} = X^4 - X^2 + \frac{1}{8}$$

and $\alpha_4 = \int_{-1}^{1} \frac{(X^4 - X^2 + \frac{1}{8})^2 dX}{\sqrt{1-X^2}} = \frac{\pi}{128}$.

The resulting polynomials are called **Chebyshev**, they also have a variety of other applications.

Example:

Repeating the previous example using Chebyshev's polynomials (rather than Legendre's) yields

$$\hat{a} = \int_{-1}^{1} \frac{\sin[\frac{\pi}{4}(1+X)]\, dX}{\sqrt{1-X^2}} \div \pi = 0.60219\,4701$$

$$\hat{b} = \int_{-1}^{1} \frac{X \sin[\frac{\pi}{4}(1+X)]\, dX}{\sqrt{1-X^2}} \div \frac{\pi}{2} = 0.51362\,51666$$

$$\begin{aligned} \hat{c} &= \int_{-1}^{1} \frac{(X^2 - \frac{1}{2}) \sin[\frac{\pi}{4}(1+X)]\, dX}{\sqrt{1-X^2}} \div \frac{\pi}{8} \\ &= -0.20709\,26885 \end{aligned}$$

$$\begin{aligned} \hat{d} &= \int_{-1}^{1} \frac{(X^3 - \frac{3}{4}X) \sin[\frac{\pi}{4}(1+X)]\, dX}{\sqrt{1-X^2}} \div \frac{\pi}{32} \\ &= -5.49281\,3693 \times 10^{-2} \end{aligned}$$

resulting in

$$\begin{aligned} &0.60219\,4701 + 0.51362\,51666\, X \\ &-0.20709\,26885(X^2 - \tfrac{1}{2}) - 0.0549281\,3693(X^3 - \tfrac{3}{4}X) \end{aligned}$$

with the typical error of

$$\sqrt{\frac{\int_{-1}^{1} \frac{\sin[\frac{\pi}{4}(1+X)]^2 dX}{\sqrt{1-X^2}} - \pi \cdot \hat{a}^2 - \frac{\pi}{2} \cdot \hat{b}^2 - \frac{\pi}{8} \cdot \hat{c}^2 - \frac{\pi}{32} \cdot \hat{d}^2}{\pi}} = 0.000964$$

(has gone up slightly, but **maximum** error is lower).

Replacing X by $\frac{4}{\pi} \cdot x - 1$ (same as before) and expanding yields:

$$-1.24477\,557 \times 10^{-3} + 1.02396\,7553x$$
$$-0.068587\,5963x^2 - 0.11337\,70686x^3$$

Laguerre Polynomials

(Just to further practise Gram_Schmidt).

Suppose the interval is $[0, \infty)$ and $W(x) = e^{-x}$. Construct the corresponding set of orthogonal polynomials, i.e.

$$\int_0^\infty e^{-x} L_i(x) L_j(x)\, dx = 0 \quad \text{whenever} \quad i \neq j.$$

$\left(\int_0^\infty e^{-x} x^n dx = n! \ \text{ may help}\right)$

$L_0(x) = 1$ with $\alpha_0 = 1$.

$L_1(x) = x + c$ (no symmetry now) where

$$\begin{aligned} \int_0^\infty e^{-x}(x+c)\, dx &= 1 + c = 0 \\ \Rightarrow c = -1 &\Rightarrow L_1(x) = x - 1 \end{aligned}$$

with $\alpha_1 = \int_0^\infty e^{-x}(x-1)^2\, dx = 1$.

$L_2(x) = x^2 + c_1 L_1(x) + c_0 L_0(x)$, where

$$\begin{aligned} \int_0^\infty e^{-x} \left(x^2 + c_1 L_1(x) + c_0 L_0(x)\right) L_0(x)\ dx &= \\ 2 + c_0 \Rightarrow c_0 &= -2 \\ \int_0^\infty e^{-x} \left(x^2 + c_1 L_1(x) + c_0 L_0(x)\right) L_1(x)\, dx &= \\ 6 - 2 + c_1 \Rightarrow c_1 &= -4 \end{aligned}$$

implying that

$$L_2(x) = x^2 - 4(x-1) - 2 = x^2 - 4x + 2$$

with $\alpha_2 = \int_0^\infty e^{-x}(x^2 - 4x + 2)^2\, dx = 4$.

$L_3(x) = x^3 + c_2 L_2(x) + c_1 L_1(x) + c_0 L_0(x)$, where

$$\int_0^\infty e^{-x}\left(x^3 + c_2 L_2 + c_1 L_1 + c_0 L_0\right) L_0 \; dx$$

$$= 6 + c_0 \Rightarrow c_0 = -6$$

$$\int_0^\infty e^{-x}\left(x^3 + c_2 L_2 + c_1 L_1 + c_0 L_0\right) L_1 \; dx$$

$$= 24 - 6 + c_1 \Rightarrow c_1 = -18$$

$$\int_0^\infty e^{-x}\left(x^3 + c_2 L_2 + c_1 L_1 + c_0 L_0\right) L_2 \; dx$$

$$= 120 - 96 + 12 + 4c_2 \Rightarrow c_2 = -9$$

resulting in

$$\begin{aligned} L_3(x) &= x^3 - 9(x^2 - 4x + 2) - 18(x - 1) - 6 \\ &= x^3 - 9x^2 + 18x - 6 \end{aligned}$$

and $\alpha_3 = \int_0^\infty e^{-x}(x^3 - 9x^2 + 18x - 6)^2 \, dx = 36$.

Hermite polynomials

Take the interval of all real numbers, with $W(x) = e^{-x^2}$.

Utilizing the $W(x) = W(-x)$ symmetry, we get:

$H_0(x) = 1$ with $\alpha_0 = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

$H_1(x) = x$ and $\alpha_1 = \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

$H_2(x) = x^2 + c$ where

$$\begin{aligned} \int_{-\infty}^{\infty} (x^2 + c)\, e^{-x^2} dx &= \frac{\sqrt{\pi}}{2} + c\sqrt{\pi} = 0 \\ &\Rightarrow c = -\frac{1}{2} \quad \Rightarrow \quad H_2(x) = x^2 - \frac{1}{2} \end{aligned}$$

and $\alpha_2 = \int_{-\infty}^{\infty} (x^2 - \frac{1}{2})^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

$H_3(x) = x^3 + c\,x$ so that

$$\begin{aligned} \int_{-\infty}^{\infty} (x^3 + c\,x)\, x\, e^{-x^2} dx &= \frac{3\sqrt{\pi}}{4} + c\frac{\sqrt{\pi}}{2} = 0 \\ &\Rightarrow c = -\frac{3}{2} \quad \Rightarrow \quad H_3(x) = x^3 - \frac{3}{2}x \end{aligned}$$

with $\alpha_3 = \int_{-\infty}^{\infty} (x^3 - \frac{3}{2}x)^2 e^{-x^2} dx = \frac{3}{4}\sqrt{\pi}$.