## INITIAL-VALUE **PROBLEM FOR (A SET OF) FIRST-ORDER ODE**

Suppose we need to solve

$$\dot{y} = f(y, t)$$

given the 'initial' value of $y$, namely $y(t_0) = y_0$. Usually, $t_0 = 0$ ($t$ represents time).

Or, more generally

$$\begin{aligned}
\dot{y}_1 &= f_1(y_1, y_2, ...y_n, t) \\
\dot{y}_2 &= f_2(y_1, y_2, ...y_n, t) \\
&\vdots \\
\dot{y}_n &= f_n(y_1, y_2, ...y_n, t)
\end{aligned}$$

given the initial value (at the *same* $t_0$) of each dependent variable.

In our discussion, we will assume that $n = 2$ (these are the easiest to visualize, when we identify $y_1$ and $y_2$ with the usual $x$ and $y$ coordinates, respectively). This is like being given an initial position of a point in 2D, and a formula (one for each component) for computing its velocity at any time, and any place, later. When the velocity is a function of position only (no $t$ dependence), the differential equations are called autonomous (we won't try to differentiate between the two, even though some techniques do).

To find a numerical solution to these, we first have to discretize time (the corresponding time step is denoted $h$), and try to find an approximate solution to $x(t_0 + h)$ and $y(t_0 + h)$. The same *step* is then repeated as many times as desired. So, it all boils down to how to advance the solution by time $h$.

The simplest (but also rather 'primitive') technique for doing this is called **Euler**; it evaluates the two velocity components at $t_0$, multiplies each of them by $h$, and add this to the existing location. This yields an exact solution only when the velocity is constant everywhere (not very likely), and has an error (of a *single* step) proportional to $h^2$ otherwise. To see that, we write (assuming we have only *one* dependent variable)

$$\begin{aligned}
y(t_0 + h) &\simeq y(t_0) + \dot{y}(t_0)h + \frac{\ddot{y}(t_0)}{2!}h^2 + \frac{\dddot{y}(t_0)}{3!}h^3 + .... \\
&= y_0 + f_0 h + \frac{f_0 f_0' + \dot{f}_0}{2}h^2 + \\
&\quad \frac{f_0^2 f_0'' + 2 f_0 \dot{f}_0' + \ddot{f}_0 + (f_0 f_0' + \dot{f}_0) f_0'}{6}h^3 + ...
\end{aligned}$$

and realize that Euler is approximating $f(t_0 + h)$ by $y_0 + f_0 h$.

To advance the solution from time $t_0$ to another (final) time $T$, the number of steps required is clearly proportional to $\frac{1}{h}$ (equal to $\frac{T - t_0}{h}$). Since the errors usually accumulate, the overall error is thus proportional to the *first* power of $h$ only - the technique is thus known as a **first-order** method. To improve its (final) accuracy by a factor of 10 (getting an extra correct digit) requires reducing $h$ ten times (10 times as much work). Due to this, the technique is hardly ever used in practice.

An improvement is the so called **modified Euler** method, in which we start by making an 'Euler' move, evaluate the velocity at the end of it, then *backtrack* to the initial location and move using the *average* of the two velocities. One can show that the error of such a move is proportional to $h^3$. But again, getting from $t_0$ to $T$ takes $\frac{T-t_0}{h}$ such steps, which 'steals' one power of $h$ from the final accuracy - this is a **second-order** technique. So now, reducing $h$ by a factor of 10 results in 2 extra correct digits.

Proof: We know that $v_1 = f_0$ and, to a sufficient accuracy

$$v_2 = f(y_0 + v_1 h, t_0 + h) = f_0 + f_0 f_0' h + \dot{f}_0 h + ...$$

This implies that

$$y_0 + \frac{v_1 + v_2}{2} h = y_0 + f_0 h + \frac{f_0 f_0' + \dot{f}_0}{2} h^2 + ...$$

A similar (second-order) technique is the **midpoint** method: with the initial velocity, we only move *half* way (multiplying it by $h/2$), evaluate a new velocity at this half-way point, backtrack, then move with this new velocity for the time $h$ (the initial velocity being discarded).

The most popular is and fourth-order technique (called **Runge-Kutta**), which works like this:

- Find the initial velocity, say $\mathbf{v}_1$
- With this velocity, move half way, and find a new velocity $\mathbf{v}_2$
- Backtrack (to the initial point), and using this new velocity, move half way *again*, finding yet another velocity $\mathbf{v}_3$
- Backtrack, and with this latest velocity, move the *full* distance, evaluating the velocity one last time (let us call it $\mathbf{v}_4$)
- Finally, backtrack one last time, and make a full move using the following **weighted average** of the four velocities

$$\frac{\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4}{6}$$

One can show (with the help of Maple) that one-step error is now proportional to $h^5$, which makes the cumulative error proportional to $h^4$ (reducing $h$ by 10, we get 4 extra digits of accuracy).

The best way to estimate the final error is to reverse the procedure (going from $T$ back to $t_0$), replacing $h$ by $-h$ (to see how close we get to $y_0$).

The easiest way (good enough for us) to deal with higher-order ODE is to make them of the first order, by doubling (second order), tripling (third), etc. their number.

EXAMPLES:

$$\ddot{y} = \frac{y \cdot \dot{y}}{1 + t^2}$$

can be converted to

$$\begin{aligned} \dot{y}_2 &= \frac{y_1 \cdot y_2}{1 + t^2} \\ \dot{y}_1 &= y_2 \end{aligned}$$

where $y_1 \equiv y$ and $y_2 \equiv \dot{y}$.

Similarly

$$\dddot{y} = \frac{y \cdot \dot{y} \cdot \ddot{y}}{1 + t^2}$$

is equivalent to

$$\begin{aligned} \dot{y}_3 &= \frac{y_1 \cdot y_2 \cdot y_3}{1 + t^2} \\ \dot{y}_2 &= y_3 \\ \dot{y}_1 &= y_2 \end{aligned}$$