SOLVING DIFFERENTIAL EQUATIONS

We will consider only equations of the following type:

$$y'' = f(x, y, y')$$

together with two *boundary conditions*, specifying the value of $y(x)$ at $x = A$ and $x = B$.

We will first consider the **Linear Case** namely:

$$y'' + p(x) \cdot y' + q(x) \cdot y = r(x)$$

where $p(x)$, $q(x)$ and $r(x)$ are specific (given) functions of x.

The equation is linear in $y(x)$ and its derivatives (y related terms have been collected on the left hand side of the equation)

This equation is quite often impossible to solve analytically, so we need a numerical technique for doing the job.

The idea: Subdivide the (A, B) interval into n equal-length subintervals of length $h = \frac{B-A}{n}$ (the nodes will be called x_0, x_1, x_2, ... x_{n-1}, x_n, with $x_0 = A$ and $x_n = B$), and set up (*ordinary*) equations for the corresponding y_0, y_1, y_2, ... y_{n-1} and y_n (the first and last are given - the rest are the unknowns). This is achieved by replacing y'' and y' by

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

and

$$\frac{y_{i+1} - y_{i-1}}{2h}$$

Note that both formulas have error terms proportional to $h^2 + ...$ The same can then be expected of our final solution.

When substituting these into the above equation, we get (point by point):

$$\begin{aligned}
&\frac{y_0 - 2y_1 + y_2}{h^2} + \frac{y_2 - y_0}{2h}\, p_1 + y_1 q_1 = r_1 \\
&\frac{y_1 - 2y_2 + y_3}{h^2} + \frac{y_3 - y_2}{2h}\, p_2 + y_2 q_2 = r_2 \\
&\vdots \\
&\frac{y_{n-2} - 2y_{n-1} + y_n}{h^2} + \frac{y_n - y_{n-2}}{2h}\, p_{n-1} + y_{n-1} q_{n-1} = r_{n-1}
\end{aligned}$$

where $p_1 \equiv p(x_1)$, $p_2 \equiv p(x_2)$, ... (similarly for q_1, q_2, ... and r_1, r_2, ...).

This can be expressed in the following matrix form:

$$\begin{matrix}
-2 + h^2 q_1 & 1 + \frac{h}{2} p_1 & 0 & \cdots & 0 \\
1 - \frac{h}{2} p_2 & -2 + h^2 q_2 & 1 + \frac{h}{2} p_2 & \cdots & \vdots \\
0 & 1 - \frac{h}{2} p_3 & -2 + h^2 q_3 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 1 - \frac{h}{2} p_{n-2} \\
0 & \cdots & 0 & 1 - \frac{h}{2} p_{n-1} & -2 + h^2 q_{n-1}
\end{matrix}$$

and the right hand side of:

$$\begin{bmatrix} r_1 h^2 - y_0\left(1 - \frac{h}{2}p_1\right) \\ r_2 h^2 \\ r_3 h^2 \\ \vdots \\ r_{n-1}h^2 - y_n\left(1 + \frac{h}{2}p_{n-1}\right) \end{bmatrix}$$

The fact that the resulting matrix of coefficients is tri-diagonal greatly simplifies the task of solving the equations.

Tri-diagonal Systems (LU Decomposition)

Any *tri-diagonal* matrix can be written as product of two matrices. The below-diagonal elements of the fist matrix are those of the original matrix, and the main-diagonal elements of the second matrix are all equal to 1.

Example:

$$\mathbb{A} = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 8 & -2 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -2 & 6 \end{bmatrix} \equiv$$

$$\begin{bmatrix} ?_{(1)} & 0 & 0 & 0 \\ -3 & ?_{(3)} & 0 & 0 \\ 0 & 1 & ?_{(5)} & 0 \\ 0 & 0 & -2 & ?_{(7)} \end{bmatrix} \begin{bmatrix} 1 & ?_{(2)} & 0 & 0 \\ 0 & 1 & ?_{(4)} & 0 \\ 0 & 0 & 1 & ?_{(6)} \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ 0 & 1 & \frac{12}{5} & 0 \\ 0 & 0 & -2 & \frac{28}{3} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This makes solving

$$\mathbb{A}\mathbf{x} = \begin{bmatrix} 7 \\ 8 \\ 2 \\ -3 \end{bmatrix}$$

a lot easier, since

$$\mathbb{L}\mathbb{U}\mathbf{x} \equiv \mathbb{L}\mathbf{y} = \begin{bmatrix} 3y_1 \\ -3y_1 + 5y_2 \\ y_2 + \frac{12}{5}y_3 \\ -2y_3 + \frac{28}{3}y_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 2 \\ -3 \end{bmatrix}$$

can be solved (rather easily) for $y_1 = \frac{7}{3}$, $y_2 = \frac{8+7}{5} = 3$, $y_3 = \frac{2-3}{\frac{12}{5}} = -\frac{5}{12}$ and $y_4 = \frac{-3-\frac{5}{6}}{\frac{28}{3}} = -\frac{23}{56}$.

Similarly, we can now deal with

$$\mathbb{U}\mathbf{x} = \begin{bmatrix} x_1 - x_2 \\ x_2 - \frac{2}{5}x_3 \\ x_3 + \frac{5}{3}x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ 3 \\ -\frac{5}{12} \\ -\frac{23}{56} \end{bmatrix}$$

from bottom up: $x_4 = -\frac{23}{56}$, $x_3 = -\frac{5}{12} + \frac{5}{3} \cdot \frac{23}{56} = \frac{15}{56}$, $x_2 = 3 + \frac{2}{5} \cdot \frac{15}{56} = \frac{87}{28}$ and $x_1 = \frac{7}{3} + \frac{87}{28} = \frac{457}{84}$.

Second example:

$$\begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 4 & -3 & -2 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 4 & -13 & 0 & 0 & 0 \\ 0 & 3 & \frac{7}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{13}{7} & 0 \\ 0 & 0 & 0 & 2 & \frac{80}{13} \end{bmatrix} \begin{bmatrix} 1 & \frac{5}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{13} & 0 & 0 \\ 0 & 0 & 1 & \frac{13}{7} & 0 \\ 0 & 0 & 0 & 1 & -\frac{14}{13} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ -3 \\ -1 \end{bmatrix}$$

yields $y_1 = 2$, $y_2 = \frac{8}{13}$, $y_3 = \frac{2}{7}$, $y_4 = \frac{23}{13}$, $y_5 = -\frac{59}{80}$ and $x_5 = -\frac{59}{80,}$ $x_4 = \frac{39}{40}$, $x_3 = -\frac{61}{40}$, $x_2 = \frac{17}{20}$, $x_1 = -\frac{1}{8}$.

We will now return to our differential equations:

Example: Solve

$$y'' = \frac{3}{1+x}\, y' + \frac{4}{(1+x)^2}\, y = 1 + x$$

subject to: $y(1) = 0$ and $y(2) = 9$ (the analytic solution is $y = x^3 + x^2 - x - 1$).

First we choose $n = 4$ ($h = \frac{1}{4}$), and build the following table:

x_i	$\frac{h\,p_i}{2}$	$h^2 q_i$	$h^2 r_i$
1.25	$-\frac{3}{18}$	$\frac{4}{81}$	$\frac{9}{64}$
1.50	$-\frac{3}{20}$	$\frac{4}{100}$	$\frac{10}{64}$
1.75	$-\frac{3}{22}$	$\frac{4}{121}$	$\frac{11}{64}$

Then, we set up the equations for y_1, y_2 and y_3:

$-\frac{158}{81}$	$\frac{15}{18}$	0	$\frac{9}{64} - 0 \times \frac{21}{18}$
$\frac{23}{20}$	$-\frac{196}{100}$	$\frac{17}{20}$	$\frac{10}{64}$
0	$\frac{25}{22}$	$-\frac{238}{121}$	$\frac{11}{64} - 9 \times \frac{19}{22}$

We already know how to solve these, the answer is:

$$\begin{aligned} y_1 &\equiv y(1.25) = 1.25876\ (1.265625) \\ y_2 &\equiv y(1.50) = 3.11518\ (3.125) \\ y_3 &\equiv y(1.75) = 5.66404\ (5.671875) \end{aligned}$$

(the errors are about 0.5, 0.3 and 0.1%, respectively).

We can improve the value of $y(1.5)$ by Richardson extrapolation, if we re-do the problem using $n = 2$ ($h = \frac{1}{2}$):

x_i	$\frac{h\,p_i}{2}$	$h^2 q_i$	$h^2 r_i$
1.50	$-\frac{3}{10}$	$\frac{4}{25}$	$\frac{10}{16}$

resulting in

$-\frac{46}{25}$	$\frac{10}{16} - 0 \times \frac{13}{10} - 9 \times \frac{7}{10}$

The right hand side has been computed based on

$$r_1 h^2 - y_0(1 - \tfrac{h}{2} p_1) - y_2(1 + \tfrac{h}{2} p_1)$$

The solution is $y_1 \equiv y(1.5) = 3.08424$, having a 1.3% error. Richardson extrapolation now yields:

$$\frac{4 \times 3.11518 - 3.08424}{3} = 3.12549$$

having the error of only 0.015% (a twenty-fold improvement over the $n = 4$ answer).

Another example:

$$y'' - \exp(\tfrac{x}{2})\, y' + \ln(1 + x)\, y = \sin x$$

with $y(0) = 2$ and $y(3) = -4$.

Here, we try to achieve better accuracy by simply increasing n. This will be done with the help of a Maple program in one of our labs.