TRIGONOMETRIC APPROXIMATION (FOURIER SERIES)

Any function can be approximated, in the $(-\pi, \pi)$ interval by a linear combination of sin and cos, thus:

$$\begin{aligned} f(x) &\simeq \frac{a_0}{2} + a_1 \cos x + b_1 \sin x \\ &+a_2 \cos 2x + b_2 \sin 2x \\ &+a_3 \cos 3x + b_3 \sin 3x + ... \end{aligned}$$

where

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\ dx \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\ dx \end{aligned}$$

This can be derived by minimizing

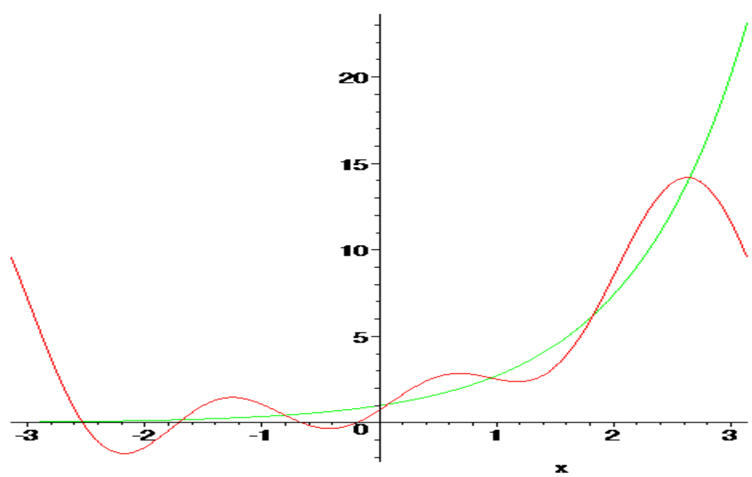
$$\int_{-\pi}^{\pi} \left[f(x) - \frac{a_0}{2} - \sum_{k=1} (a_k \cos kx + b_k \sin kx) \right]^2 dx$$

which is made easy by the fact that the set of $\cos kx$ and $\sin kx$ functions is *orthogonal*.

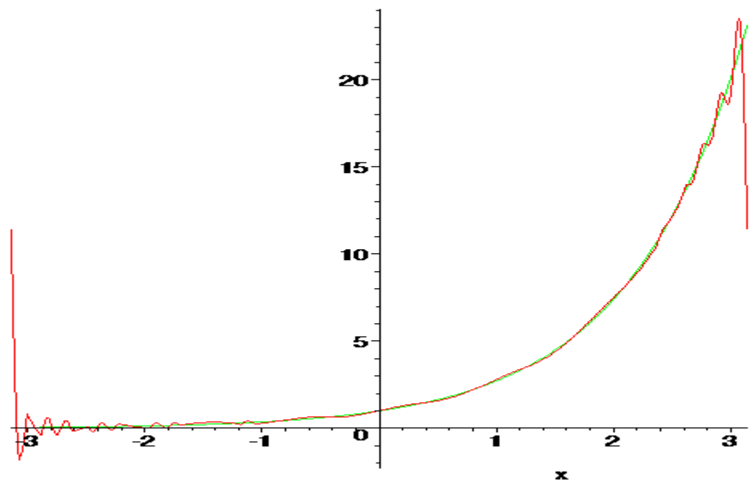
EXAMPLE:

$$\begin{aligned} e^x &\simeq 3.67608 - 3.67608 \cos x + 3.67608 \sin x \\ &+1.47043 \cos 2x - 2.94086 \sin 2x \\ &-0.735216 \cos 3x + 2.20565 \sin 3x \end{aligned}$$

which, when displayed graphically, yields the following (far from spectacular) fit:

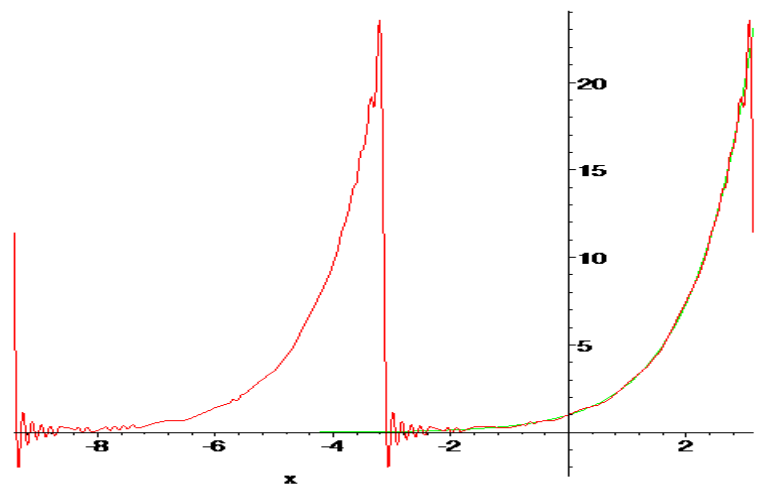


When increasing the number of terms to 81 (up to $\cos 40x$ and $\sin 40x$), we get



(note the so called Gibbs phenomenon).

Note that the trigonometric expansion is a periodic function:



If we want to fit a function by a combination of $\sin$ and $\cos$ in a different *finite* interval, say (A, B), we first *shift* the function to $(-\pi, \pi)$ by introducing a new independent variable

$$X \equiv \frac{2x - (A+B)}{B-A} \cdot \pi$$

or, in reverse

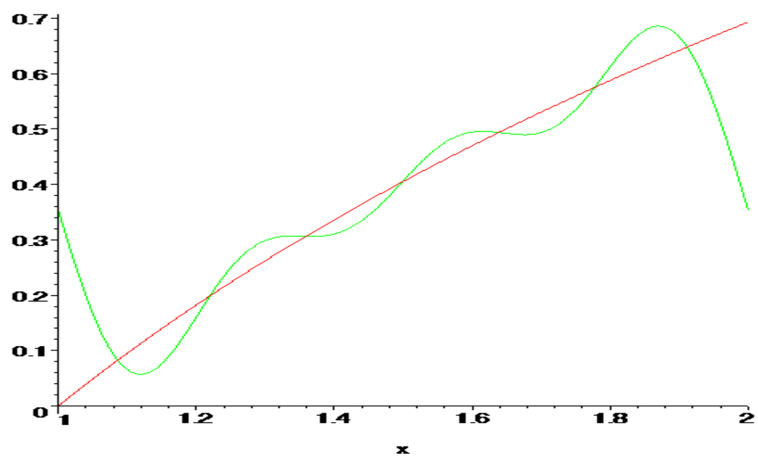
$$x = \frac{B-A}{2\pi} \cdot X + \frac{A+B}{2}$$

do the same fit as before in terms of X, and then shift back to x.

EXAMPLE: This time, we consider the $\ln x$ function in the $(1,2)$ interval, which implies $x = \frac{X}{2\pi} + \frac{3}{2}$. Fitting $\ln\left(\frac{X}{2\pi} + \frac{3}{2}\right)$ over the $(-\pi, \pi)$ interval yields

$$\begin{aligned}&3.67608 + 0.023558 \cos X + 0.215401 \sin X\\ &-0.00620228 \cos 2X - 0.109594 \sin 2X\\ &+0.00278771 \cos 3X + 0.0733257 \sin 3X\end{aligned}$$

Replace X by $(2x-3)\cdot\pi$, and we have our 'approximation' to $\ln x$ over $(1,2)$, which looks like this:



DISCRETE VERSION

Let us go back to $(-\pi, \pi)$. We divide the interval into $2m$ subintervals of equal length, denoting the end points x_0, x_1, x_2, ..., x_{2m}. We then find the trigonometric approximation by minimizing

$$\sum_{j=0}^{2m-1}\left[f(x_j) - \frac{a_0}{2} - a_n \cos nx_j \right.$$
$$\left. - \sum_{k=1}^{n-1}(a_k \cos kx_j + b_k \sin kx_j)\right]^2$$

Note that $n < m$.

Since the $\cos kx_j$ and $\sin kx_j$ functions are still orthogonal, this time in the following, discrete sense

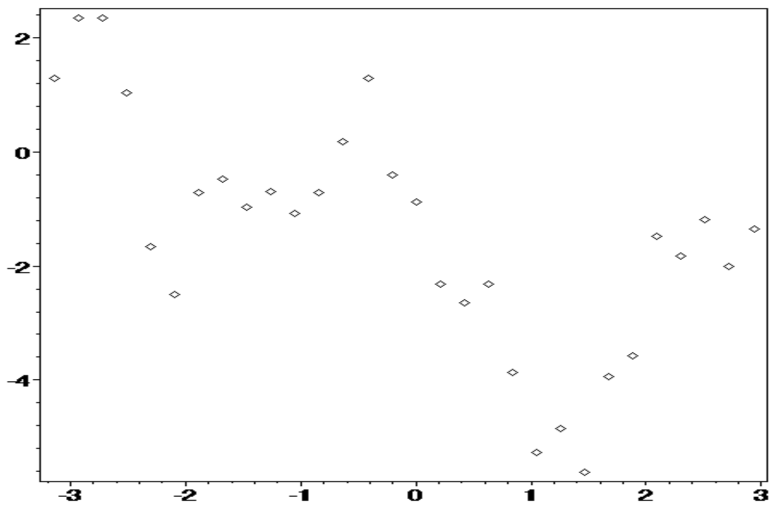
$$\sum_{j=0}^{2m-1} \cos kx_j \sin \ell x_j = 0$$

the above minimization has the following simple solution

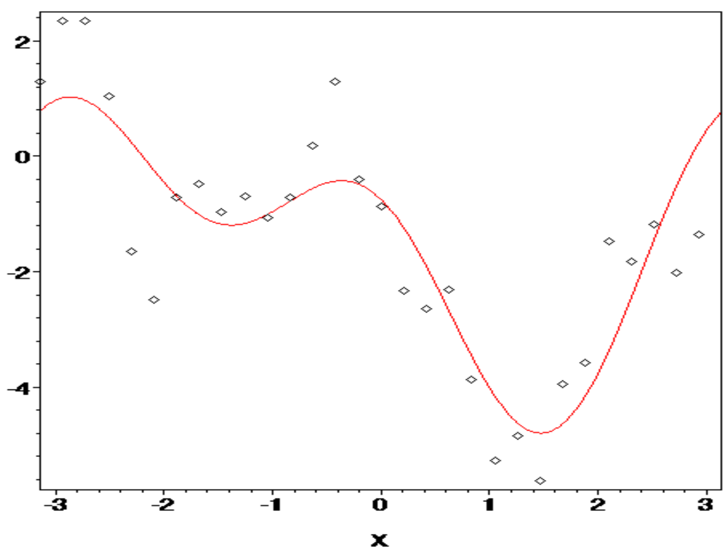
$$a_k = \frac{1}{m}\sum_{j=0}^{2m-1} f(x_j)\cos(kx_j)$$
$$b_k = \frac{1}{m}\sum_{j=0}^{2m-1} f(x_j)\sin(kx_j)$$

EXAMPLE:

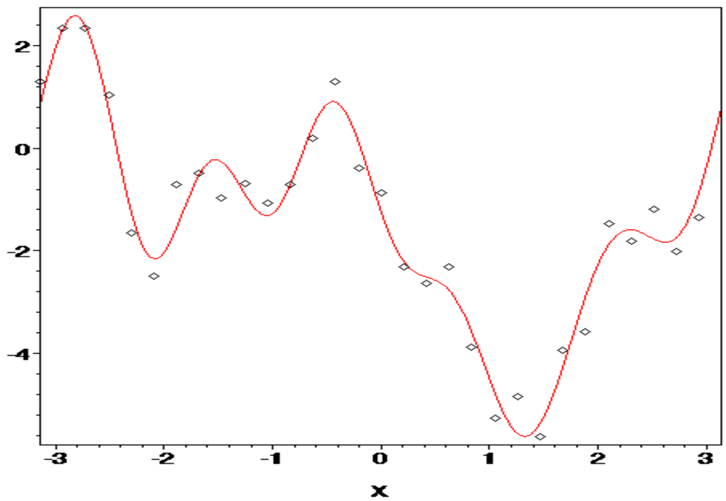
Let $m = 15$ and our y values are:



With $n = 2$, we get $-1.460 - 0.764\cos(X) - 1.817\sin(X) + 1.479\cos(2X)$



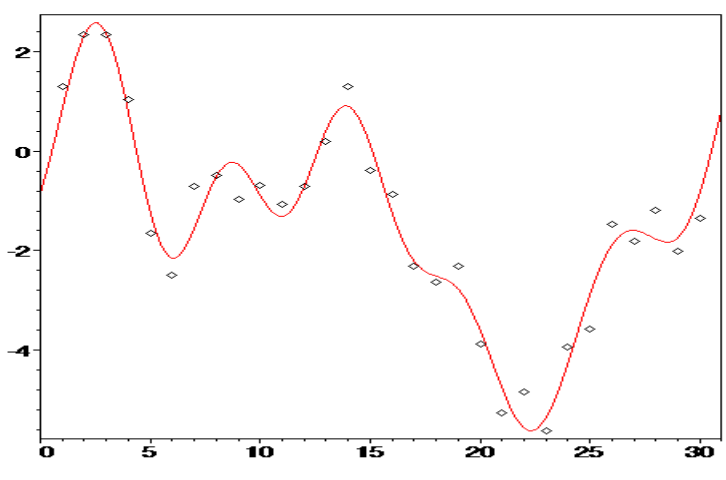
Similarly, for $n = 6$, the results are $-1.460 - 0.764\cos(X) - 1.817\sin(X) + 1.479\cos(2X) - 0.515\sin(2X) - 0.009\cos(3X) - 0.388\sin(3X) + 0.077\cos(4X) + 0.562\sin(4X) - 0.277\cos(5X) - 0.896\sin(5X) - 0.291\cos(6X)$ and



We can then easily change to x scale to go from A (for the first point) to B (for the last), by

$$X = \frac{x - A}{B - A} \cdot \pi \left(2 - \frac{1}{m}\right) - \pi$$

getting

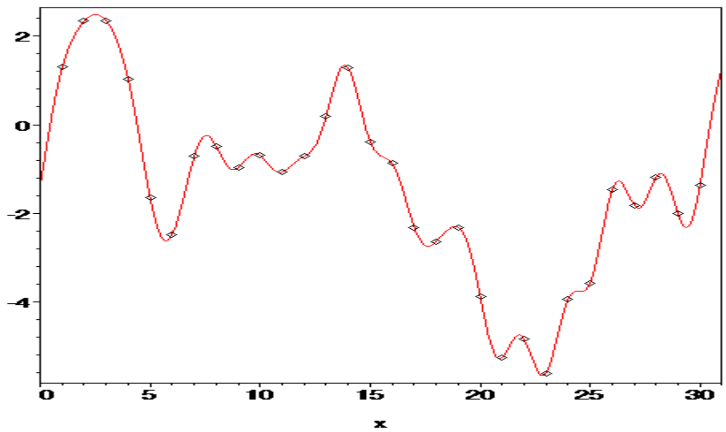


INTERPOLATING TRIGONOMETRIC POLYNOMIAL

This is the previous case with $n = m$. Clearly, the individual errors at x_0, x_1, x_2, ..x_{2m-1} must now be all equal to zero. Also, the solution now changes to

$$\frac{a_0}{2} + \frac{a_m}{2}\cos mx_j + \sum_{k=1}^{m-1} a_k \cos kx_j + b_k \sin kx_j$$

(the rest being the same). For our previous data we get (after the same re-scaling):



When m is equal to a (large) power of 2, then there is an algorithm which can compute the a_k and b_k coefficients while minimizing the number of additions and (mainly) multiplications. It is called **fast Fourier transform**, and it is extremely complicated; I'll try to demonstrate how it works in one of our labs. The number of multiplications is reduced from $(2m)^2$ of the regular technique, to $3m + m\log_2 m$.

Thus, for example, when $m = 2^{20} = 1,048,576$, the regular technique would require $2^{42} = 4,398,046,511,104$ multiplications (not feasible), against $2^{20}(3+20) = 24,117,248$ required by FFT (piece of cake).

APPENDIX

$$x_j = -\pi + \frac{2\pi}{2m}j \qquad j = 0..2m-1$$

$$\begin{aligned}
\sum_{j=0}^{2m-1} \sin(Kx_j)\cos(Lx_j) &= 0 \\
\sum_{j=0}^{2m-1} \sin(Kx_j)\sin(Lx_j) &= 0 \\
\sum_{j=0}^{2m-1} \cos(Kx_j)\cos(Lx_j) &= 0
\end{aligned}$$

for any $0 \le K \le m$ and $0 \le L \le m$, with the exception of the last two, when $K = L$.

First we show

$$\sum_{j=0}^{2m-1} \exp(iMx_j) = 0$$

when $0 < M < 2m$. Same as

$$\exp(-i\pi M)\left[1 + a + a^2 + ... + a^{2m-1}\right] = (-1)^M \frac{1 - a^{2m}}{1 - a}$$

where

$$a = \exp(iM\frac{\pi}{m})$$

Note that

$$a^{2m} = \exp(iM \cdot 2\pi) = 1$$

which proves the above.

Also note that, when $M = 0$, the same sum is trivially equal to $2m$, and when $M = 2m$, we get $\frac{0}{0}$, so by L'Hopital rule, the sum equals

$$\lim_{M \to 2m} \frac{1 - \exp(iM \cdot 2\pi)}{1 - \exp(iM\frac{\pi}{m})} = \frac{-2\pi i}{-i\frac{\pi}{m}} = 2m$$

as well (the result is always *real*).

Now, we have to do is this:

$$\sum_{j=0}^{2m-1} \exp(iKx_j)\exp(iLx_j) = \sum_{j=0}^{2m-1} \exp[i(K+L)x_j]$$

$$= \sum_{j=0}^{2m-1} \left[\cos(Kx_j) + i\sin(Kx_j)\right]\left[\cos(Lx_j) + i\sin(Lx_j)\right]$$

$$= \sum_{j=0}^{2m-1} \cos(Kx_j)\cos(Lx_j) - \sum_{j=0}^{2m-1} \sin(Kx_j)\sin(Lx_j)$$

$$+i\sum_{j=0}^{2m-1} \sin(Kx_j)\cos(Lx_j) + i\sum_{j=0}^{2m-1} \cos(Kx_j)\sin(Lx_j)$$

and

$$
\begin{aligned}
&\sum_{j=0}^{2m-1} \exp(iKx_j)\exp(-iLx_j) = \sum_{j=0}^{2m-1} \exp[i(K-L)x_j] \\
&= \sum_{j=0}^{2m-1} [\cos(Kx_j) + i\sin(Kx_j)]\,[\cos(Lx_j) - i\sin(Lx_j)] \\
&= \sum_{j=0}^{2m-1} \cos(Kx_j)\cos(Lx_j) + \sum_{j=0}^{2m-1} \sin(Kx_j)\sin(Lx_j) \\
&+i\sum_{j=0}^{2m-1} \sin(Kx_j)\cos(Lx_j) - i\sum_{j=0}^{2m-1} \cos(Kx_j)\sin(Lx_j)
\end{aligned}
$$

The result is thus non-zero only when $K = L$ ($K = L = m$ and $K = L = 0$ get double contribution).