Gaussian Integration of $\int_{-1}^{1} y(x)\,dx$

Any three (four, five)-node formula will be exact for all quadratic (cubic, quartic, ...) polynomials, but if we are lucky (e.g. Simpson's rule), we can go higher than this. How high can we go?

The answer: If we choose n nodes to be the roots of the n^{th} degree Legendre polynomial (they all must be in the -1 to 1 range), the corresponding rule will be exact for all polynomials of degree up to and including $2n-1$ (instead of the usual $n-1$). This also pushes the order of the leading error term from h^{n+1} to h^{2n+1} - a very substantial improvement!

Proof:

For any polynomial, say $q(x)$ of degree $n-1$ (or less)

$$\int_{-1}^{1} q(x)\cdot\phi_n(x)\,dx = 0$$

since q can be expressed as a liner combination of $\phi_0, \phi_1, ...\phi_{n-1}$.

Let $p(x)$ be an arbitrary polynomial of degree smaller than $2n$. Then:

$$p(x) = q(x)\cdot\phi_n(x) + r(x)$$

where both q and r are polynomials of degree $n-1$ or less.

Our Gaussian rule will correctly (i.e. exactly) integrate both the first term (equal to zero) and the second term (any n-point formula does). It thus correctly integrates $p(x)$.

Example:

To derive a 3-point Gaussian formula, we must first find the roots of $\phi_3(x) = x^3 - \frac{3}{5}x$, namely $x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$ and $x_3 = \sqrt{\frac{3}{5}}$.

We know how to continue:

$$\int_{-1}^{1} y(x)\,dx \approx 2\cdot[c_1\,y(x_1) + c_2\,y(x_2) + c_1\,y(x_3)]$$

making it exact for $y=1$ and $y=x^2$, i.e.

$$\begin{aligned} 4c_1 + 2c_2 &= 2 \\ 4c_1\cdot\frac{3}{5} &= \frac{2}{3} \end{aligned}$$

Solution: $c_1 = \frac{5}{18}$ and $c_2 = \frac{8}{18}$.

The final rule:

$$\int_{-1}^{1} y(x)\,dx \approx 2\cdot\frac{5\,y(x_1) + 8\,y(x_2) + 5\,y(x_3)}{18}$$

It is automatically correct also for $y = x^4$ (and all odd powers of x):

$$2 \times 5 \times \frac{9}{25} \div 9 = \frac{2}{5}$$

It can be transformed to work with any limits:

$$\int_A^B y(x)\,dx \simeq (B-A) \cdot$$
$$\frac{5\,y(\frac{A+B}{2} - \sqrt{\frac{3}{5}}\frac{B-A}{2}) + 8\,y(\frac{A+B}{2}) + 5\,y(\frac{A+B}{2} + \sqrt{\frac{3}{5}}\frac{B-A}{2})}{18}$$

Applied to $\int_0^{\pi/2} \sin x\,dx$, it yields:

$$\frac{\pi}{36}\left(5\sin[\tfrac{\pi}{4}(1-\sqrt{\tfrac{3}{5}})] + 8\sin(\tfrac{\pi}{4}) + 5\sin[\tfrac{\pi}{4}(1+\sqrt{\tfrac{3}{5}})]\right)$$
$$= 1.00000\,8122$$

a spectacular improvement over Simpson's (also 3 point) result of 1.0022.

Furthermore, applying it separately to the $(0, \frac{\pi}{4})$ and $(\frac{\pi}{4}, \frac{\pi}{2})$ subintervals (the composite-rule idea), results in

$$\frac{\pi}{72}\left(5\sin[\tfrac{\pi}{8}(1-\sqrt{\tfrac{3}{5}})] + 8\sin(\tfrac{\pi}{8}) + 5\sin[\tfrac{\pi}{8}(1+\sqrt{\tfrac{3}{5}})]\right)$$
$$+\frac{\pi}{72}\left(5\sin[\tfrac{\pi}{8}(3-\sqrt{\tfrac{3}{5}})] + 8\sin(\tfrac{3\pi}{8}) + 5\sin[\tfrac{\pi}{8}(3+\sqrt{\tfrac{3}{5}})]\right)$$
$$= 1.00000\,0119$$

a 68 fold increase in accuracy (theory predicts 64).

Romberg algorithm further improves the two results to $(64 \times 1.00000\,0119 - 1.00000\,8122) \div 63 = 0.99999\,9992$

Integrals With Weights

such as, for example, $\int_0^\infty e^{-x} \cdot y(x)\,dx$. The idea is the same (use the roots of Laguerre, instead of Legendre, polynomials).

Example 1:

For a 3 point rule, we first solve (with the help of Maple)

$$x^3 - 9x^2 + 18x - 6 = 0$$

The roots are:

$$\begin{aligned} x_1 &= 0.4157745568 \\ x_2 &= 2.294280360 \\ x_3 &= 6.289945083 \end{aligned}$$

Replacing $y(x)$ by

$$\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}\,y(x_1)+\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}\,y(x_2)+\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}\,y(x_3)$$

we multiply by e^{-x} and integrate:

$$0.7110930101\,y_1+0.2785177336\,y_2+0.0103892565\,y_3$$

The rule is exact not only for $y(x)=1$, x and x^2:

$$0.7110930101+0.2785177336+0.0103892565$$
$$=1\equiv\int\limits_0^\infty e^{-x}dx$$

$$0.7110930101x_1+0.2785177336x_2+0.0103892565x_3$$
$$=1\equiv\int\limits_0^\infty x\,e^{-x}dx$$

$$0.7110930101x_1^2+0.2785177336\,x_2^2+0.0103892565x_3^2$$
$$=2\equiv\int\limits_0^\infty x^2e^{-x}dx$$

but also for $y(x)=x^3$, x^4 and x^5:

$$0.7110930101x_1^3+0.2785177336\,x_2^3+0.0103892565\,x_3^3$$
$$=6\equiv\int\limits_0^\infty x^3e^{-x}dx$$
$$0.7110930101x_1^4+0.2785177336x_2^4+0.0103892565x_3^4$$
$$=24\equiv\int\limits_0^\infty x^4\,e^{-x}dx$$
$$0.7110930101x_1^5+0.2785177336x_2^5+0.0103892565x_3^5$$
$$=120\equiv\int\limits_0^\infty x^5e^{-x}dx$$

For $y(x)=x^6$ we get:

$$0.7110930101x_1^6+0.2785177336x_2^6+0.0103892565x_3^6$$
$$=684$$

(5% off the correct answer of 720).

Applying the formula to $\int_0^\infty \frac{\exp(-x)}{x+2}\,dx$ yields

$$\frac{0.\,7110930101}{x_1+2}+\frac{0.2785177336}{x_2+2}+\frac{0.0103892565}{x_3+2}=0.3605$$

It should now be quite obvious how to construct a Gaussian formula with a given number of nodes to approximate

$$\int_{-1}^{1} \frac{y(x)}{\sqrt{1-x^2}}\, dx$$

using the roots of the corresponding Chebyshev polynomial.

Example 2:

2-point Gaussian formula for $\int_0^1 \frac{y(x)}{\sqrt{x}}\, dx$.

Since the corresponding $\phi_2(x)$ is now given by $x^2 - \frac{6}{7}x + \frac{3}{35}$ (Assignment 4), our two nodes are:

$$\begin{aligned} x_1 &= \frac{3}{7} - \sqrt{\frac{9}{49} - \frac{3}{35}} = \frac{3}{7} - \frac{2}{35}\sqrt{30} \\ x_2 &= \frac{3}{7} + \sqrt{\frac{9}{49} - \frac{3}{35}} = \frac{3}{7} + \frac{2}{35}\sqrt{30} \end{aligned}$$

Replacing $y(x)$ by $\dfrac{x - x_2}{x_1 - x_2}\, y(x_1) + \dfrac{x - x_1}{x_2 - x_1}\, y(x_2)$, dividing by $\sqrt{x}$ and integrating yields:

$$\int_0^1 \frac{y(x)}{\sqrt{x}}\, dx \approx \left(1 + \tfrac{\sqrt{30}}{18}\right)\, y(x_1) + \left(1 - \tfrac{\sqrt{30}}{18}\right)\, y(x_2)$$

The rule is correct for $y(x) = 1$, x, x^2 and x^3:

$$\begin{aligned} \left(1 + \tfrac{\sqrt{30}}{18}\right) + \left(1 - \tfrac{\sqrt{30}}{18}\right) &= 2 \\ \left(1 + \tfrac{\sqrt{30}}{18}\right) x_1 + \left(1 - \tfrac{\sqrt{30}}{18}\right) x_2 &= \frac{2}{3} \\ \left(1 + \tfrac{\sqrt{30}}{18}\right) x_1^2 + \left(1 - \tfrac{\sqrt{30}}{18}\right) x_2^2 &= \frac{2}{5} \\ \left(1 + \tfrac{\sqrt{30}}{18}\right) x_1^3 + \left(1 - \tfrac{\sqrt{30}}{18}\right) x_2^3 &= \frac{2}{7} \end{aligned}$$

but

$$\left(1 + \tfrac{\sqrt{30}}{18}\right) x_1^4 + \left(1 - \tfrac{\sqrt{30}}{18}\right) x_2^4 = \frac{258}{1225} \neq \frac{2}{9}$$

(off by about 5%).

To approximate

$$\int_0^1 \frac{\exp(x^2)}{\sqrt{x}}\, dx \simeq$$

$$\left(1 + \tfrac{\sqrt{30}}{18}\right) \exp(x_1^2) + \left(1 - \tfrac{\sqrt{30}}{18}\right) \exp(x_2^2) = 2.528$$

reasonably close (0.6% error) to the exact answer of 2.543.