NONLINEAR EQUATIONS

We start with equation for one unknown:

$$f(x) = 0$$

We can find a solution (solutions) graphically, but only to two significant digits. We can then improve the accuracy by

Newton's Method:

Fit a straight line with a slope of $f'(x_0)$ trough the point $[x_0,\ f(x_0)]$:

$$y - f(x_0) = f'(x_0) \cdot (x - x_0)$$

then find its intercept with the x axis, by solving

$$-f(x_0) = f'(x_0) \cdot (x - x_0)$$

This yields:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

which is a better, but not necessarily 10-digit accurate solution.

But, we can apply the same idea again, using x_1 in place of x_0. This will result in

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

And again:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

etc., until the numbers no longer change.

One can show that this procedure is *quadratically convergent*, meaning that the number of correct digits roughly doubles in each **iteration** (one step of the procedure).

Example: Solve

$$e^x = 2 - x$$

First we plot $f(x) = e^x - 2 + x$. The graph intersects the x-axis at, roughly, $x_0 = 1$. Then, compute $f'(x) = e^x + 1$ and we are ready to iterate:

$$\begin{aligned}
x_1 &= 1 - \frac{e - 2 + 1}{e + 1} = 0.53788\,28428 \\
x_2 &= x_1 - \frac{e^{x_1} - 2 + x_1}{e^{x_1} + 1} = 0.\mathbf{44}561\,67486 \\
x_3 &= x_2 - \frac{e^{x_2} - 2 + x_2}{e^{x_2} + 1} = 0.\mathbf{44285}\,67246 \\
x_4 &= x_3 - \frac{e^{x_3} - 2 + x_3}{e^{x_3} + 1} = 0.\mathbf{44285\,44011} \\
x_5 &= x_4 - \frac{e^{x_4} - 2 + x_4}{e^{x_4} + 1} = 0.\mathbf{44285\,44011}
\end{aligned}$$

Quadratic convergence is clearly observed.

The method fails (convergence becomes extremely slow) when $f(x)$ only touches the x axis, without crossing it (this is an indication that both $f(x)$ *and* $f'(x)$ have a root at that point).

Example: We know that $f(x) = 1+\sin x$ has a root at $x = \frac{3}{2}\pi = 4.71238\,8981$ ($\equiv 270^o$). If we try to find it by the regular technique (starting at $x_0 = 5$), we get

$$\begin{aligned}
x_1 &= x_0 - \frac{1+\sin x_0}{\cos x_0} = \mathbf{4}.85519\,4921 \\
x_2 &= x_1 - \frac{1+\sin x_1}{\cos x_1} = \mathbf{4.7}8367\,0356 \\
x_3 &= x_2 - \frac{1+\sin x_2}{\cos x_2} = \mathbf{4.7}4801\,457 \\
x_4 &= x_3 - \frac{1+\sin x_3}{\cos x_3} = \mathbf{4.7}3019\,9891 \\
x_5 &= x_4 - \frac{1+\sin x_4}{\cos x_4} = \mathbf{4.7}2129\,4199 \\
x_6 &= x_5 - \frac{1+\sin x_5}{\cos x_5} = \mathbf{4.71}684\,156 \\
x_7 &= x_6 - \frac{1+\sin x_6}{\cos x_6} = \mathbf{4.71}461\,527
\end{aligned}$$

If instead we solve $(1+\sin x)' = \cos x = 0$, we get

$$x_1 = x_0 + \frac{\cos x_0}{\sin x_0} = \mathbf{4.7}0418\,7084$$

$$x_2 = x_1 + \frac{\cos x_1}{\sin x_1} = \mathbf{4.71238}\,9164$$

$$x_3 = x_2 + \frac{\cos x_2}{\sin x_2} = \mathbf{4.71238\,898}$$

$$x_4 = x_3 + \frac{\cos x_3}{\sin x_3} = \mathbf{4.71238\,898}$$

One can now easily verify that

$$1 + \sin 4.71238\,898 = 0$$

(Regular) Example: When dealing with the third-degree Laguerre polynomial, we had to rely on Maple to get its three roots. Now, we can do this ourselves. Plotting

$$x^3 - 9x^2 + 18x - 6 = 0$$

indicates that there is a root near $x_0 = 6$. We thus get

$$\begin{aligned}
x_1 &= x_0 - \frac{x_0^3 - 9x_0^2 + 18x_0 - 6}{3x_0^2 - 18x_0 + 18} = \mathbf{6}.33333\,3333 \\
x_2 &= x_1 - \frac{x_1^3 - 9x_1^2 + 18x_1 - 6}{3x_1^2 - 18x_1 + 18} = \mathbf{6}.\mathbf{29}071\,5373 \\
x_3 &= x_2 - \frac{x_2^3 - 9x_2^2 + 18x_2 - 6}{3x_2^2 - 18x_2 + 18} = \mathbf{6}.\mathbf{28994\,5}332 \\
x_4 &= x_3 - \frac{x_3^3 - 9x_3^2 + 18x_3 - 6}{3x_3^2 - 18x_3 + 18} = \mathbf{6}.\mathbf{28994\,5083}
\end{aligned}$$

Once we have a root of a cubic equation, we can *deflate* the polynomial by carrying out the following *synthetic division*:

$$\begin{aligned}
&(x^3 - 9x^2 + 18x - 6) \div (x - 6.28994\,5083) = \\
&x^2 - 2.710054917\,x + 0.9539034002
\end{aligned}$$

The remaining two roots can then be found by the usual formula:

$$\begin{aligned}
&\tfrac{2.710054917}{2} \mp \sqrt{(\tfrac{2.710054917}{2})^2 - 0.9539034002} \\
=\ & 0.41577\,45565 \quad \text{and} \quad 2.29428\,0362
\end{aligned}$$

Several Unknowns

First two nonlinear equations for two unknowns:

$$\begin{aligned}
F_1(x_1, x_2) &= 0 \\
F_2(x_1, x_2) &= 0
\end{aligned}$$

Finding a reasonably accurate starting (*initial*) solution is now a lot more difficult - we will assume that a reasonably good estimate is provided to us, we will call them x_{1_0} and x_{2_0}.

F_1 and F_2 can be (Taylor) expanded, around this point, as follows:

$$\begin{aligned}
&F_1(x_1, x_2) = F_1(x_{1_0}, x_{2_0}) + \\
&\frac{\partial F_1(x_{1_0}, x_{2_0})}{\partial x_1}(x_1 - x_{1_0}) + \frac{\partial F_1(x_{1_0}, x_{2_0})}{\partial x_2}(x_2 - x_{2_0}) + ... \\
&F_2(x_1, x_2) = F_2(x_{1_0}, x_{2_0}) + \\
&\frac{\partial F_2(x_{1_0}, x_{2_0})}{\partial x_1}(x_1 - x_{1_0}) + \frac{\partial F_2(x_{1_0}, x_{2_0})}{\partial x_2}(x_2 - x_{2_0}) + ...
\end{aligned}$$

or, in matrix notation,

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{x}_0) + \left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0} (\mathbf{x} - \mathbf{x}_0) + ...$$

where $\mathbf{x}_0$ is a column vector with components x_{1_0} and x_{2_0}, and $\frac{\partial \mathbf{F}}{\partial \mathbf{x}}$ denotes, the following *Jacobian*

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} \equiv \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}$$

Setting to $\mathbf{0}$ and solving for $\mathbf{x}$:

$$\mathbf{x}_1 = \mathbf{x}_0 - \left[\left. \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right|_{\mathbf{x}_0} \right]^{-1} \mathbf{F}(\mathbf{x}_0)$$

(one iteration). In this spirit we can continue:

$$\mathbf{x}_2 = \mathbf{x}_1 - \left[\left. \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \right|_{\mathbf{x}_1} \right]^{-1} \mathbf{F}(\mathbf{x}_1)$$

etc., until convergence reached.

Example: Solve

$$\begin{aligned} x_1 \cos x_2 + 0.716 &= 0 \\ x_2 \sin x_1 - x_1^2 - 1.305 &= 0 \end{aligned}$$

starting with $x_{1_0} = 1$ and $x_{2_0} = 3$.

The Jacobian:

$$\begin{bmatrix} \cos x_2 & -x_1 \sin x_2 \\ x_2 \cos x_1 - 2x_1 & \sin x_1 \end{bmatrix}$$

Evaluating the left hand side of each of our equations (using the initial values of x_1 and x_2) yields

$$\mathbf{F}_0 = \begin{bmatrix} -0.2739924966 \\ 0.2194129540 \end{bmatrix}$$

Similarly evaluating the Jacobian:

$$\mathbb{J}_0 = \begin{bmatrix} -0.9899924966 & -0.1411200081 \\ -0.379093082 & 0.8414709848 \end{bmatrix}$$

This implies:

$$\mathbf{x}_1 = \mathbf{x}_0 - [\mathbb{J}_0]^{-1} \mathbf{F}_0 = \begin{bmatrix} 0.7748646744 \\ 2.637824472 \end{bmatrix}$$

Now, repeat the process (best done with Maple):

```
F := [ x[1]*cos(x[2]) + 0.716,
    x[2]*sin(x[1]) - x[1]^2 - 1.305 ];
```

$J := \textbf{matrix}(2, 2)$:

for i **to** 2 **do** **for** j **to** 2 **do**

$J[i, j] := \textbf{diff}(F[i], x[j])$ **end do** **end do:**

$x := [1., 3.]$:

with(linalg):

$x := \textbf{evalm}(x - \textbf{linsolve}(J, F))$;

By re-executing the last line, we will automatically get $\mathbf{x}_1$, $\mathbf{x}_2$, etc.:

[0.**7**748646744, **2**.637824472]

[0.**78**25429930, **2**, **71**9825617]

[0.**78546**82773, **2**.**7178**42406]

[0.**785460622**0, **2**.**717875728**]

[0.**7854606228**, **2**.**717875728**].

It should be clear how the formulas extend to the case of three or more unknowns.

Example: Solve:

$$
\begin{aligned}
\frac{x_1 + x_2 + x_3}{3} &= 7 \\
\sqrt[3]{x_1 x_2 x_3} &= 4 \\
\frac{3}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}} &= \frac{16}{7}
\end{aligned}
$$

For the initial values, take $x_1 = 1.5$, $x_2 = 5$ and $x_3 = 10$.

In the previous computer program, we change the definition of F to:

$F := [\ x[1] + x[2] + x[3] - 21,$

$x[1] * x[2] * x[3] - 64,$

$1/x[1] + 1/x[2] + 1/x[3] - 21/16\ ]$;

increase the dimensions (from 2 to 3, in the next two lines), and change the initial values.

As result, we get:

[.6989495801, 3.572619045, 16.72843138]

[.8746180084, 4.895879953, 15.22950204]

[.9791494539, 3.948814285, 16.07203626]

[.9992737556, 4.004281052, 15.99644519]

[.9999992176, 3.999999364, 16.00000142]

[.9999999991, 4.000000005, 16.00000000]

(verify against the original equations).

If we used the equations in their original form, it would have taken us 9 iterations to reach the same conclusion.

The choice of the initial values is quite critical, see what would happen if we change x_{1_0} to 2.0 (don't forget to type '**restart**;' first).