BROCK UNIVERSITY

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Course: MATH 3P85 | Number of students: 19
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Time of Examination: 19:00-22:00 | Instructor: J. Vrbik

Open-book exam. Full credit given for 18 (out of 32) complete, correct and properly simplified answers; these **must** be entered in your booklet (do it as soon as you compute them); e-mail your Maple to **jvrbik@brocku.ca** (keep a copy).

1. Consider the following game: a gambler pays \$4 to be dealt a random hand of 7 cards; he then collects \$4 for each ace in his hand and \$1 for each spade (\$5 for the ace of spaces). Compute

   (a) the expected value of his net win in one round of this game, and the corresponding standard deviation,

   (b) the probability that he wins, in one round, more than \$2 (hint: you will need the corresponding PGF of his net winnings),

   (c) the probability that he wins more than \$8 (in *total*) in 40 independent rounds of this game (full credit given for either the exact answer, or the answer computed using Normal approximation - extra marks earned when you give both),

   (d) the probability that he wins money in 5 rounds, loses money in 6, and breaks even in 1, assuming that this time he plays 12 independent rounds of the game.

2. Consider the following bivariate pdf of two random variables X and Y

$$f(x,y) = c\ (x+y) \qquad \text{when} \quad 0 < x < y < 1$$

   (zero otherwise). Find

   (a) the value of c,

   (b) the marginal pdf of X and the corresponding support,

   (c) $\text{Var}(3X - 4 \mid Y = 0.5)$, hint: first find the corresponding conditional pdf,

   (d) the pdf of $U = X + Y$ and the corresponding support.

3. Consider a random independent sample (RIS from now on) of size 5 from an Exponential distribution with a mean of β. Find

   (a)
$$\Pr\left(2(X_1 + X_2) > X_4 + X_4 + X_3\right)$$
   (hint: remember that a sum of independent exponentials has gamma distribution),

   (b) the pdf of $Y = \exp\left(-\frac{\sum_{i=1}^{5} X_i}{\beta}\right)$ and its support (hint: ditto),

   (c) the pdf of $U = X_{(1)}$ and its support - identify the resulting distribution.

4. Consider a random variable X having the following pdf

$$f(x) = \frac{1}{x^2} \qquad \text{when } x > 1$$

(zero otherwise). Find

(a) the *distribution*'s median and quartile deviation,

(b) the probability that the *sample* median of a RIS of size 63 from this distribution is bigger than 2.3 (here again, full credit given for either the exact or the Normal-approximation answer - extra marks earned when you give both),

(c) the probability that the sample *mid-range* value (of the same RIS) is bigger than 20 (hint: compute the probability of $X_{(1)} + X_{(63)} > 40$),

(d) the pdf of $Y = X_1 + X_2$, where X_1 and X_2 are the first *two* values of the same sample (hint: use *convolution*) - do not forget to quote the resulting support.

5. Consider a RIS of size 13 from $N(\mu = -3, \sigma = 2.7)$.. Compute the following probabilities

(a)

$$\Pr\left(5 \cdot \left(\bar{X} + 3\right)^2 > s^2\right)$$

(b)

$$\Pr\left(\bar{X}^2 > 6\right)$$

(c)

$$\Pr\left(\sum_{i=1}^{6} X_i^2 - \frac{\left(\sum_{i=1}^{6} X_i\right)^2}{6} > 20\right)$$

(d)

$$\Pr\left(\sum_{i=1}^{6} X_i^2 - \frac{\left(\sum_{i=1}^{6} X_i\right)^2}{6} > \sum_{i=7}^{13} X_i^2 - \frac{\left(\sum_{i=7}^{13} X_i\right)^2}{7}\right)$$

where $\bar{X}$ and s^2 are the sample mean and sample variance, respectively, and X_1, $X_2, \ldots X_{13}$ are the individual sample values.

6. Having a RIS of size n from a distribution with the following pdf

$$f(x) = \left(1 - \frac{x}{\theta}\right)^{\theta - 1} \qquad \text{when} \qquad 0 < x < \theta$$

(zero otherwise), where $\theta > 0$ (hint: when integrating, Maple needs to *assume* this), find

(a) the corresponding method-of-moments estimator of θ

(b) and its asymptotic variance.

7. Having a RIS of size n from a distribution with the following pdf

$$f(x) = \frac{\lambda \cdot \theta^{\lambda}}{x^{\lambda+1}} \qquad \text{when} \qquad x > \theta$$

(zero otherwise), where $\theta > 0$ and $\lambda > 0$, find

   (a) two sufficient statistics for estimating θ and λ,

   (b) maximum-likelihood estimators of θ and λ.

8. Let X and Y have the bivariate Normal distribution with the following parameters: $\mu_x = 16$, $\mu_y = -203$, $\sigma_x = 3.4$, $\sigma_y = 23$ and $\rho = 0.84$. Find

   (a)
$$\Pr\left(9X - \frac{Y}{2} > 250\right)$$

   (b)
$$\Pr\left(9X > 84 \mid Y = -243\right)$$

   (c) the joint moment generating function of X and Y (express it in terms of variables t_1 and t_2),

   (d)
$$\mathbb{E}\left(X^2 Y\right)$$

   hint: use the previous MGF.

9. Having a RIS of size n from a distribution with the following pdf

$$f(x) = \frac{\exp\left(-\dfrac{(\theta - \ln x)^2}{2\lambda^2}\right)}{\lambda\, x\sqrt{2\pi}} \qquad \text{when} \qquad x > 0$$

(zero otherwise), where $\lambda > 0$, find

   (a) two sufficient statistics for estimating θ and λ,

   (b) maximum-likelihood estimators of θ and λ,

   (c) the asymptotic (Rao-Cramer) variance-covariance matrix of the two ML estimators.

10. Consider rolling a pair of dice (considered as *one* roll) repeatedly until getting the third occurrence of ‘snake’s eyes’ (each of the two dice showing one dot). Find

   (a) the expected number of such rolls and the corresponding standard deviation,

   (b) the probability that this will take at least 150 rolls.