BROCK UNIVERSITY

Final Examination: December 2015	Number of Pages: 3
Course: MATH 3F85	Number of students: 18
Date of Examination: Dec. 12, 2015	Number of Hours: 3
Time of Examination: 8:00-11:00	Instructor: J. Vrbik

This is an open-book exam. Full credit will be given for 20 (out of 35) correct (to at least 4 significant digits) and complete answers. **Numbers** must be presented in their *decimal* form (simple *fractions* are acceptable as well); they should not contain π, $\sqrt{..}$, unevaluated functions e.g. arctan(2), etc. Students are allowed to use *basic* Maple **only** (no internet searching, no Maple 'packages' such as 'with(Statistics)', etc.). All answers **must** be entered in the examination booklet (rough work and your Maple may be attached).

1. Consider two random variables X and Y having the following joint PDF (probability density function)

$$f(x,y) = \begin{cases} \frac{108}{25}(x+y^2)\exp(-x-2y) & 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

Find (with each PDF answer, specify the corresponding *support*),

(a) the marginal PDF of X,

(b) the conditional PDF of Y given that $X = 2$,

(c) the expected value and standard deviation of X,

(d) the covariance between X and Y,

(e) the PDF of $V = \exp(-X)$,

(f) $\Pr(X + Y < 2)$,

(g) the PDF of $U = X + Y$.

2. Consider a random variable X with the following PDF

$$f(x) = \begin{cases} (1+x)/2 & -1 < x \le 0 \\ 1/2 & 0 < x \le 1 \\ (2-x)/2 & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the corresponding

(a) distribution function (CDF),

(b) expected value and standard deviation,

(c) median and quartile deviation,

(d) $\Pr(X < 0.65)$.

Assuming we take a RIS (random independent sample) of size 51 from this distribution and using the appropriate Normal approximation (different in each case!) compute the

(e) $\Pr(\tilde{X} < 0.65)$, where $\tilde{X}$ is the sample *median*,

(f) $\Pr(\bar{X} < 0.65)$, where $\bar{X}$ is the sample *mean*.

3. Consider a distribution with the following PDF

$$f(x) = \frac{8x^4}{3\sqrt{\pi}\theta^{5/2}} \exp(-\frac{x^2}{\theta}) \qquad \text{when } x > 0, \qquad f(x) = 0 \quad \text{otherwise}$$

where θ is a positive parameter whose exact value is unknown. Assuming that we use a RIS of size n from this distribution to estimate θ, find

   (a) the corresponding Cramer-Rao variance,
   (b) a sufficient *statistic* for such an estimation,
   (c) the sufficient *unbiased estimator* of θ
   (d) and its efficiency.

4. Two random variables X and Y have the bivariate Normal distribution with $\mu_X = 26$, $\mu_Y = -3.7$, $\sigma_X = 4.1$, $\sigma_Y = 1.1$ and $\rho = -0.79$. Compute

   (a) the expected value and standard deviation of $2X - 3Y$,
   (b) $\text{Cov}(2X - 3Y, 3X - 2Y)$,
   (c) $\Pr(X > 27)$,
   (d) $\Pr(X > 27 \mid Y = -2.9)$.

5. Consider a RIS of size 9 from the beta(2,3) distribution. Compute

   (a) the expected value and standard deviation of the sample *mean*,
   (b) the expected value and standard deviation of the sample *median* (find exact answers, do not use the large-n approximation!),
   (c) the probability that the *smallest* of the 9 observations is bigger than 0.1,
   (d) the probability that the difference between the largest observation and the smallest observation is bigger than 0.7 .

6. Assume that customers arrive at a specific store randomly and independently of each other, at a constant average rate of 17.3 customers per hour. We start observing (and counting) the arriving customers at 9:30 in the morning. Compute

   (a) the probability that we will have seen at least 10 customers enter the store by 10:00 (hint: use the Poisson distribution),
   (b) the probability that we have to wait more than 15 minutes (till after 9:45) for the arrival of the third customer,
   (c) the expected time (use the 11:45:52 format for the answer, to the nearest second) of the third arrival (after 9:30) and the corresponding standard deviation (given in minutes and seconds). Hint: use the gamma distribution.

7. Taking a RIS of size 32 from a Normal distribution with the mean of 12.4 and the standard deviation of 3.7, find the probability that ($\bar{X}$ and s are the corresponding sample mean and the sample standard deviation, respectively):

   (a) $12 < \bar{X} < 13$,

   (b) $3.5 < s < 4$,

   (c) $12 < \bar{X} < 13 \cap 3.5 < s < 4$,

   (d) $\bar{X} < 12.4 - \frac{s}{4}$.

8. Consider a distribution with the following PDF

$$f(x) = \frac{x^4}{24\theta^5} \exp(-\frac{x}{\theta}) \qquad \text{when } x > 0, \qquad f(x) = 0 \quad \text{otherwise}$$

   where θ is a positive parameter whose exact value is unknown. Assuming that we use a RIS of size n to estimate θ, find

   (a) the corresponding Cramer-Rao variance,

   (b) maximum likelihood estimator of θ

   (c) and its expected value and efficiency.