1.

$$f(x,y) = \frac{108}{25}(x+y^2)\exp(-x-2y) \qquad\qquad 0<x<y$$

(a)

$$f_X(x) = \int_x^\infty f(x,y)dy = \frac{27}{25}(2x^2+4x+1)\exp(-3x) \qquad\qquad x>0$$

(b)

$$\frac{f(2,y)}{f_X(2)} = \frac{4}{17}(y^2+2)\exp(4-2y) \qquad\qquad y>2$$

(c)

$$\begin{aligned}
\mu_X &= \int_0^\infty x\cdot f_X(x)dx = \frac{3}{5} \\
\sigma_X &= \sqrt{\int_0^\infty (x-\mu_X)^2\cdot f_X(x)\ dx} = 0.5033
\end{aligned}$$

(d)

$$\begin{aligned}
\mu_Y &= \int_0^\infty y\int_0^y f(x,y)dx\ dy \\
\mathrm{Cov}(x,y) &= \int_0^\infty (y-\mu_Y)\int_0^y (x-\mu_X)f(x,y)dx\ dy = 0.1653
\end{aligned}$$

(e)

$$\begin{aligned}
x &= -\ln v \\
\left|\frac{dx}{dv}\right| &= \frac{1}{v} \\
f_V(v) &= \frac{27}{25}\left(2(\ln v)^2+4\ln v+1\right)v^2 \qquad\qquad 0<v<1
\end{aligned}$$

(f)

$$\int_0^1\int_0^{2-x} f(x,y)dy\ dx = 0.5067$$

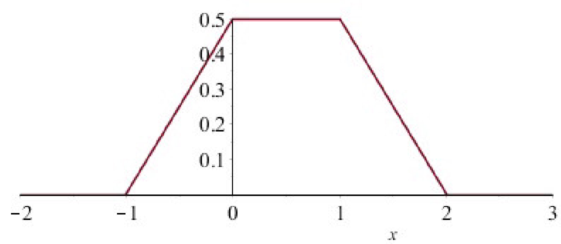
(g)

$$F_U(u) = \int_0^{u/2}\int_0^{u-x} f(x,y)dy\ dx$$
$$f_U(u) = \tfrac{dF_U(u)}{du}$$
$$= -\tfrac{108}{25}(u+1)^2\exp(-2u) + \tfrac{27u^2+162u+108}{25}\exp(-\tfrac{3}{2}u) \qquad u>0$$

2.

$$f(x)=\begin{cases}0 & x<-1\\ \frac{x+1}{2} & -1\le x<0\\ \frac{1}{2} & 0\le x<1\\ \frac{2-x}{2} & 1\le x<2\\ 0 & 2\le x\end{cases}$$

It looks like this:



(a)

$$F(x)=\begin{cases}0 & x<-1\\ \frac{(x+1)^2}{4} & -1\le x<0\\ \frac{x}{2}+\frac{1}{4} & 0\le x<1\\ 1-\frac{(2-x)^2}{4} & 1\le x<2\\ 1 & 2\le x\end{cases}$$

(b)

$$\begin{aligned}\mu &= \int_{-1}^{0} x\cdot\tfrac{x+1}{2}dx+\int_0^1 \tfrac{x}{2}dx+\int_1^2 x\cdot\tfrac{2-x}{2}dx=\frac{1}{2}\\ \mathbb{E}\left(X^2\right) &= \int_{-1}^{0} x^2\cdot\tfrac{x+1}{2}dx+\int_0^1 \tfrac{x^2}{2}dx+\int_1^2 x^2\cdot\tfrac{2-x}{2}dx=\frac{2}{3}\\ \sigma &= \sqrt{\tfrac{2}{3}-\tfrac{1}{4}}=\sqrt{\tfrac{5}{12}}=0.6455\end{aligned}$$

(c) It is obvious that $F(0)=\frac{1}{4}$, $F(\frac{1}{2})=\frac{1}{2}$ and $F(1)=\frac{3}{4}$ which imply that $\tilde{\mu}=\frac{1}{2}$ and $\tilde{\sigma}=\frac{1-0}{2}=\frac{1}{2}$.

(d)

$$F(0.65)=\tfrac{0.65}{2}+\tfrac{1}{4}=0.5750$$

(e) Since $\tilde{X}$ has, approximately, the $\mathcal{N}(\tilde{\mu},\frac{1}{2f(\tilde{\mu})\sqrt{51}})\equiv\mathcal{N}(\frac{1}{2},\frac{1}{\sqrt{51}})$ distribution

$$\Pr(\tilde{X}<0.65)\simeq\sqrt{\tfrac{51}{2\pi}}\int_{-\infty}^{0.65}\exp(-\tfrac{(\tilde{x}-1/2)^2\cdot 51}{2})d\tilde{x}=0.8580$$

(f) Similarly, since $\bar{X}$ has, approximately, the $\mathcal{N}(\mu,\frac{\sigma}{\sqrt{51}})$ distribution

$$\Pr(\bar{X}<0.65)\simeq\sqrt{\tfrac{51}{2\pi\sigma^2}}\int_{-\infty}^{0.65}\exp(-\tfrac{(\bar{x}-1/2)^2\cdot 51}{2\sigma^2})d\bar{x}=0.9515$$

3.

$$f(x) = \frac{8x^4 \exp(-\frac{x^2}{\theta})}{3\sqrt{\pi}\theta^{5/2}} \qquad x > 0$$

(a) Since

$$\frac{\partial^2 \ln f(x)}{\partial \theta^2} = \frac{5}{2\theta^2} - \frac{2x^2}{\theta^3}$$

and

$$\mathbb{E}\left(X^2\right) = \int_0^\infty x^2 f(x) dx = \frac{5}{2}\theta$$

$$RCV = \frac{2\theta^2}{5n}$$

(b) Since

$$\sum_{i=1}^n \ln f(x_i) = n \ln \tfrac{8}{3\sqrt{\pi}} - \frac{\sum_{i=1}^n x_i^2}{\theta} + 4\sum_{i=1}^n \ln x_i - \frac{5n}{2} \ln \theta$$

$\sum_{i=1}^n X_i^2$ is a sufficient statistic for estimating θ.

(c) Since (as we already know)

$$\mathbb{E}\left(\frac{\sum_{i=1}^n X_i^2}{n}\right) = \mathbb{E}\left(X^2\right) = \frac{5}{2}\theta$$

it is obvious that $\frac{2\sum_{i=1}^n X_i^2}{5n}$ is an unbiased estimator of θ.

(d) Since

$$\mathbb{E}\left(X^4\right) = \int_0^\infty x^4 f(x) dx = \frac{35}{4}\theta^2$$

i.e.

$$\mathrm{Var}(X^2) = \frac{10}{4}\theta^2$$

which further implies that

$$\mathrm{Var}(\tfrac{2}{5}\overline{X^2}) = \left(\frac{2}{5}\right)^2 \cdot \frac{10}{4n}\theta^2 = \frac{2\theta^2}{5n}$$

making the previous estimator 100% efficient.

4. We are given $\mu_X = 26$, $\sigma_X = 4.1$, $\mu_Y = -3.7$, $\sigma_Y = 1.1$ and $\rho = -0.79$.

(a)

$$\begin{aligned} 2\mu_X - 3\mu_Y &= 63.10 \\ \sqrt{2^2\sigma_X^2 + (-3)^2\sigma_Y^2 + 2 \cdot 2 \cdot (-3)\sigma_X\sigma_Y\rho} &= 10.99 \end{aligned}$$

(b)

$$\begin{aligned} & 2\cdot 3\mathrm{Var}(x)+2\cdot(-2)\mathrm{Cov}(X,Y)+(-3)\cdot 3\mathrm{Cov}(X,Y)+(-3)\cdot(-2)\mathrm{Var}(Y) \\ = \quad & 6\sigma_X^2-11\sigma_X\sigma_Y\rho+6\sigma_Y^2=154.4 \end{aligned}$$

(c)

$$\sqrt{\tfrac{1}{2\pi\sigma_X^2}}\int_{27}^{\infty}\exp\left(-\tfrac{(x-\mu_X)^2}{2\sigma_X^2}\right)dx=0.4037$$

(d) The conditional distribution of X is Normal with

$$\begin{aligned} \mu_{X|Y} &= \mu_X+\rho\sigma_X\cdot\tfrac{-2.9-\mu_Y}{\sigma_Y}=23.6444 \\ \sigma_{X|Y}^2 &= \sigma_X^2\cdot(1-\rho^2)=6.31888 \end{aligned}$$

The answer is:

$$\sqrt{\tfrac{1}{2\pi\sigma_{X|Y}^2}}\int_{27}^{\infty}\exp\left(-\tfrac{(x-\mu_{X|Y})^2}{2\sigma_{X|Y}^2}\right)dx=0.09095$$

5. We know that

$$\begin{aligned} f(x) &= 12x(1-x)^2 \qquad 0<x<1 \\ F(x) &= \int_0^x f(u)du=3x^4-8x^3+6x^2 \qquad 0<x<1 \end{aligned}$$

(a) Using formulas for the expected value and variance of `beta` distribution, we get

$$\begin{aligned} \frac{2}{2+3} &= \frac{2}{5} \\ \sqrt{\frac{2\cdot 3}{(2+3)^2(2+3+1)\cdot 9}} &= \frac{1}{15}=0.06667 \end{aligned}$$

(the variance of the sample mean had to be further divided by $n=9$).

(b)

$$\begin{aligned} \frac{9!}{4!\cdot 4!}\int_0^1 x\cdot F(x)^4\left(1-F(x)\right)^4 f(x) &= 0.3885028 \\ \sqrt{\frac{9!}{4!\cdot 4!}\int_0^1 (x-0.3885028)^2\cdot F(x)^4\left((1-F(x)\right)^4 f(x)} &= 0.09075 \end{aligned}$$

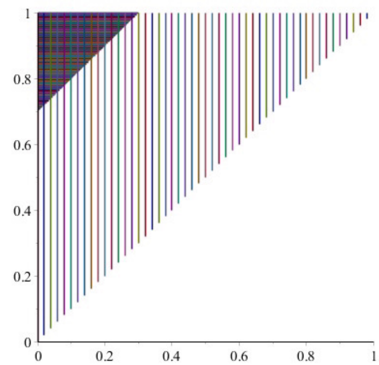
(c) This happens iff all observations are bigger than 0.1; answer:

$$\left(1-F(0.1)\right)^9=0.9477^9=0.6166$$

(d) Based on the joint PDF of the smallest and biggest observation, we get

$$9 \cdot 8 \int_{0.7}^{1} \int_{0}^{y-0.7} \big(F(y) - F(x)\big)^7 f(x) f(y) \; dx \; dy = 0.1747$$

This is the region over which we have to integrate



6. We are given $\lambda = 17.3$ arrivals per hour.

(a) Since $\Lambda_a = \lambda \cdot 0.5 = 8.75$, we get

$$1 - \exp(-\Lambda_a) \sum_{i=0}^{9} \frac{\Lambda_a^i}{i!} = 0.3666$$

(b) This is the same as getting fewer than 3 arrivals during the next 15 minutes. So now $\Lambda_b = \lambda \cdot 0.25 = 4.375$, and the answer is

$$\exp(-\Lambda_b) \sum_{i=0}^{2} \frac{\Lambda_b^i}{i!} = 0.1942$$

(c) Since the average inter-arrival times are equal to $\frac{1}{\lambda}$ hours, the expected time of the third arrival (from 'now') is

$$\frac{3}{\lambda} \cdot 60 = 10.4046 \qquad \textit{minutes}$$

This translates to 9:40:24.

Similarly, the corresponding standard deviation is

$$\sqrt{\frac{3}{\lambda^2}} \cdot 60 = 6.0071 \qquad \textit{minutes}$$

i.e. 6 minutes and 0.4 seconds.

7. This time we know that $\mu = 12.4$, $\sigma = 3.7$ and $n = 32$.

   (a)
   $$\sqrt{\frac{n}{2\pi\sigma^2}} \int_{12}^{13} \exp\left(-\frac{(x-\mu)^2 \cdot n}{2\sigma^2}\right) dx = 0.55010$$

   (b)
   $$\begin{aligned} \Pr(3.5 &< s < 4) = \Pr\left(\frac{(n-1)\cdot 3.5^2}{\sigma^2} < \frac{(n-1)\cdot s^2}{\sigma^2} < \frac{(n-1)\cdot 4^2}{\sigma^2}\right) \\ &= \Pr(27.7392 < \chi^2_{31} < 36.2308) \\ &= \frac{\int_{27.7392}^{36.2308} u^{\frac{31}{2}-1} \exp(-\frac{u}{2})\ du}{\Gamma(\frac{31}{2}) \cdot 2^{\frac{31}{2}}} = 0.39695 \end{aligned}$$

   (c) Since $\bar{X}$ and s are *independent* RVs, all we have to do is to *multiply* the previous two answers, getting
   $$0.55010 \cdot 0.39695 = 0.2184$$

   (d)
   $$\begin{aligned} \Pr(\bar{X} &< 12.4 - \tfrac{s}{4}) = \Pr\left(\frac{\bar{X}-12.4}{\frac{s}{\sqrt{32}}} < -\frac{\sqrt{32}}{4}\right) = \Pr\left(\mathrm{t}_{31} < -\sqrt{2}\right) \\ &= \frac{\Gamma(\frac{32}{2})}{\Gamma(\frac{31}{2})\sqrt{31\pi}} \int_{-\infty}^{-\sqrt{2}} (1 + \tfrac{y^2}{31})^{-\frac{32}{2}} dy = 0.08363 \end{aligned}$$

8.
$$f(x) = \frac{x^4 \exp(-\frac{x}{\theta})}{24\theta^5} \qquad x > 0$$

   (a) Since
   $$\frac{\partial^2 \ln f(x)}{\partial\theta^2} = \frac{5}{\theta^2} - \frac{2X}{\theta^3}$$
   and
   $$\mathbb{E}\left(X\right) = \int_0^\infty x \cdot f(x) dx = 5\theta$$
   $$RCV = \frac{\theta^2}{5n}$$

   (b)
   $$\sum_{i=1}^n \frac{\partial \ln f(x_i)}{\partial\theta} = \frac{\sum_{i=1}^n x_i}{\theta^2} - \frac{5n}{\theta} = 0$$
   implies that the MLE of θ is
   $$\frac{\sum_{i=1}^n X_i}{5n}$$

(c) Since the expected value of X is 5θ (as we already know), this estimator is fully unbiased. Computing

$$\mathbb{E}\left(X^2\right) = \int_0^\infty x^2 \cdot f(x)dx = 30\theta^2$$

results in $\mathrm{Var}(X) = 5\theta^2$, which further implies that

$$\mathrm{Var}\left(\frac{\sum_{i=1}^n X_i}{5n}\right) = \frac{\mathrm{Var}(X)}{25n} = \frac{\theta^2}{5n}$$

This agrees with CRV; the estimator is thus 100% efficient.