1.

(a)

$$f(u) = \tfrac{1}{2}\int_0^y e^{-x-\frac{y-x}{2}}dx = e^{-\frac{y}{2}} - e^{-y} \qquad y > 0$$

(b)

$$1 - F_V(v) = \Pr(X > v \cap Y > v) = e^{-v} \cdot e^{-\frac{v}{2}} = e^{-\frac{3}{2}v} \qquad v > 0$$

implying

$$f(v) = \tfrac{3}{2}e^{-\frac{3}{2}v} \qquad v > 0$$

which is Exponential with the mean of $\frac{2}{3}$.

2.

(a)

$$\begin{aligned} \mu_{X-Y} &= 15 - (-4) = 19 \\ \sigma^2_{X-Y} &= 3^2 + 2^2 - 2 \cdot 3 \cdot 2 \cdot (-\tfrac{7}{9}) = \frac{67}{3} \end{aligned}$$

Answer:

$$\tfrac{1}{\sqrt{2\pi}}\int_{\frac{20-19}{\sqrt{67/3}}}^{\infty} \exp(-z^2/2)dz = 41.62\%$$

(b)

$$\begin{aligned} \mu_{X|Y} &= 15 - \tfrac{7}{9} \cdot 3 \cdot \tfrac{-21/10+4}{2} = \tfrac{767}{60} \\ \sigma^2_{X|Y} &= 3^2\left(1 - (-\tfrac{7}{9})^2\right) = \tfrac{32}{9} \end{aligned}$$

Answer:

$$\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{12.3-767/60}{\sqrt{32/9}}} \exp(-z^2/2)dz = 39.88\%$$

3.

(a)

$$\Pr\left(|\bar{X}-20| < \frac{s}{3}\right) = \Pr\left(\left|\tfrac{\bar{X}-20}{s/\sqrt{12}}\right| < \tfrac{\sqrt{12}}{3}\right) = \Pr\left(|\mathrm{t}_{11}| < \tfrac{\sqrt{12}}{3}\right)$$

$$= \tfrac{\Gamma(\frac{12}{2})}{\Gamma(\frac{11}{2})\sqrt{11\pi}} \int_{-\frac{\sqrt{12}}{3}}^{\frac{\sqrt{12}}{3}} \left(1+\tfrac{y^2}{11}\right)^{-12/2} dy = 72.73\%$$

(b)

$$\Pr(s > 4) = \Pr\left(\tfrac{11s^2}{3.7^2} > \tfrac{1100\cdot 4^2}{37^2}\right) = \Pr\left(\chi^2_{11} > \tfrac{17600}{1369}\right)$$

$$= \tfrac{1}{\Gamma(\frac{11}{2})\cdot 2^{11/2}} \int_{\frac{17600}{1369}}^{\infty} y^{11/2-1} e^{-y/2} dy = 30.28\%$$

4.

(a)

$$\Pr(\overline{X} > 6) = \Pr\left(\frac{\sum_{i=1}^{5} X_i}{5} > 6\right) = \Pr(\mathsf{gamma}(5,8) > 30)$$

$$= \tfrac{1}{4!\cdot 8^5} \int_{30}^{\infty} y^4 e^{-y/8} dy = 67.75\%$$

(b)

$$\Pr\{X_1+X_2+X_3 > 2X_4+2X_5\} = \Pr\left(\frac{X_4+X_5}{\sum_{i=1}^{5} X_i} < \frac{1}{3}\right)$$

$$= \Pr\left(\mathsf{beta}(2,3) < \frac{1}{3}\right) = \tfrac{\Gamma(5)}{\Gamma(2)\Gamma(3)} \int_0^{1/3} y(1-y)^2 dy = 40.74\%$$

5.

$$f_Y(y) = \int_0^{1-y} (10x^2+4xy)dx = \frac{10}{3} - 8y + 6y^2 - \frac{4}{3}y^3 \qquad 0 < y < 1$$

(a)

$$f_{X|Y}(x) = \left.\frac{10x^2+4xy}{\frac{10}{3} - 8y + 6y^2 - \frac{4}{3}y^3}\right|_{y=1/3} = \frac{27}{26}x + \frac{405}{52}x^2 \qquad 0 < x < \frac{2}{3}$$

(b)

$$\mu_Y = \int_0^1 y\left(\frac{10}{3} - 8y + 6y^2 - \frac{4}{3}y^3\right) dy = 0.2333$$

Solving

$$\int_0^u f_Y(y)dy = \frac{10}{3}u - 4u^2 + 2u^3 - \frac{1}{3}u^4 = \frac{1}{2}$$

for u in the $(0, 1)$ range yields the median of 0.1889.