BROCK UNIVERSITY

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Course: MATH 4F21	Number of students: 2
Date of Examination: Apr. 11, 2001	Number of Hours: 3
Time of Examination: 14:00-17:00	Instructor: J. Vrbik

This is an open-book exam. Full credit given for **7** correct and complete answers.

1. Consider a simple Markov chain with the following transition probability matrix:

$$\mathbb{P} = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$$

Assuming that X_0 is generated from the following distribution

$X_0 =$	1	2	3
Pr	0.6	0.0	0.4

find:

(a) $\Pr\{X_2 = 3 \mid X_4 = 1\}$,

(b) $\Pr\{X_{1000} = 2\}$,

(c) the expected number of transitions it will take to enter, for the first time, State 2, and the corresponding standard deviation.

2. Consider a branching process with 3 initial members (Generation 0), and the following probability generating function for the number of offsprings

$$F(s) = \left(\frac{0.71 - 0.11s}{1 - 0.4s} \right)^2$$

Compute:

(a) probability that the last 'surviving' generation (with at least one member) is Generation 4.

(b) expected value and standard deviation of the number of members of Generation 5,

(c) expected value and standard deviation of total progeny.

3. Find the expected value and the corresponding standard deviation of the number of rolls of a die to generate the pattern $EE6EE6$ for the first time (6 means a six, E implies anything else). What is the probability that this will take fewer than 20 rolls.

4. Consider an $M/G/\infty$ queue where service times are uniformly distributed between 15 and 45 minutes, and customers arrive at the rate of 13.2 per hour. Find:

   (a) the long-run proportion of time with more than 5 customers in the system.

   (b) the probability that, 35 minutes after starting the operation (in State 0), at least two satisfied customers have already finished their service and left.

5. Consider a Birth and Death process with $\lambda_n = 7.4 \times n$ per hour, $\mu_n = 7.7 \times n$ per hour, and the $X(0) = 3$. Compute:

   (a) expected value and standard deviation of $X(10 \text{ min.})$,

   (b) probability that $X(10 \text{ min.}) > 2$,

   (c) expected time till extinction.

6. Solve

$$2z^2 \overset{\bullet}{P}(z,t) = z\,P'(z,t) + 2\,P(z,t)$$

   subject to the following initial condition:

$$P(z,0) = 1$$

7. For a continuous-time Markov process with the following (per hour) rates

$$\begin{bmatrix} \times & 1.4 & 1.9 \\ 1.2 & \times & 1.6 \\ 1.0 & 1.5 & \times \end{bmatrix}$$

   (a) find the corresponding stationary distribution,

   (b) if the process is currently in State 2, what is the probability that the process will be, 15 minutes later, in State 3?

8. Consider a Brownian motion with no drift and the diffusion coefficient of $31.7\frac{\text{m}^2}{\text{hour}}$. Starting at 8:00 am. in the initial location of 13.9 m, what is the probability the process will

   (a) have a negative value at 11:00 am.

   (b) avoid the zero value until 11:00 am.

   (c) return to 13.9 m, at least once, between 11:00 and 12:15.

9. Find $\exp(\frac{\mathbb{P}}{5})$, where

$$\mathbb{P} = \begin{bmatrix} -3 & 1 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -4 \end{bmatrix}$$

10. Find the first four (up to and including ρ_4) serial correlation coefficients of the following autoregressive model

$$X_i = 2.7\,X_{i-1} - 2.59\,X_{i-2} + 0.873\,X_{i-3} + \epsilon_i$$

where $\epsilon_i \in \mathcal{N}(0,2)$. Also, what is the stationary variance of X_i, and the value of the partial correlation coefficient $\rho(X_i, X_{i-2}|\,X_{i-1})$? Is the process stable?

11. Find (in terms of exact fractions) the fixed vector of the following transition probability matrix:

$$\mathbb{P} = \begin{bmatrix} 0 & 0.3 & 0.4 & 0 & 0.3 \\ 0.4 & 0 & 0 & 0.6 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 \\ 0 & 0.5 & 0.1 & 0 & 0.4 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{bmatrix}$$

If the corresponding Markov chain starts in State 3, what is the probability that 1001 transitions later it will be in State 4? What is the long-run percentage of time spent in State 4? Is this Markov chain reversible (usually, one can get the answer by constructing only a single element of $\overset{\vee}{\mathbb{P}}$)?

12. Solve the following difference equation

$$2a_{i+1} - 5a_i + 2a_{i-1} = \frac{9\,i}{2^i}$$

knowing that $a_1 = -3$ (don't confuse it with a_0) and $a_{13} = -\frac{9}{256}$.