BROCK UNIVERSITY

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This is an open-book exam. Full credit given for **7** correct and complete answers.

1. Consider a simple Markov chain with the following transition probability matrix:

$$\mathbb{P} = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$$

   Find:

   (a) $\Pr\{X_2 = 3 \cap X_4 = 1\}$ assuming that X_0 is generated from the following distribution

$X_0 =$	1	2	3
Pr	0.3	0.3	0.4

   (b) the stationary probability vector,

   (c) Probability of visiting State 1 before State 2.

2. Consider a branching process with the following distribution for the number of offsprings

$X =$	0	1	2	3
Pr	0.31	0.34	0.25	0.10

   and 4 initial members (Generation 0). Compute:

   (a) probability of extinction within the first five generations,

   (b) probability of ultimate extinction,

   (c) expected value and standard deviation of the number of members of Generation 5.

3. Find the expected value and the corresponding standard deviation of the number of rolls of a die to generate the pattern $6EEEE6$ for the first time (6 means a six, E implies anything else).

4. Consider an $M/G/\infty$ queue where service times are of random duration, having the following distribution function

$$G(t) = 1 - (1 + \frac{t}{\beta}) \exp(-\frac{t}{\beta})$$

   (recognize this as the $\mathsf{gamma}(2, \beta)$ distribution; take $\beta = 12$ min.), and customers arrive at the rate of 13.2 per hour. Find:

(a) the long-run proportion of time with more than 5 customers being serviced,

(b) the probability that more than 5 customers will be serviced half an hour after opening, starting with no customers.
Hint: $\int(1+\frac{u}{\beta})\exp(-\frac{u}{\beta})\,du = -(2\beta+u)\exp(-\frac{u}{\beta})$

5. Consider an $M/M/\infty$ queue with customers arriving at the rate of 13.2 per hour and the average service time of 24 minutes. Also assume that, at the opening time, there are three customers already waiting for service (this defines the initial state). Compute:

   (a) the expected value and the corresponding standard deviation of the number of customers in being serviced half an hour after opening,

   (b) the long-run proportion of time with more than 5 customers.

6. Derive the stationary distribution for an $M/M/1$ queue with a finite waiting room of size $N-1$ (which means that it can accommodate only $N-1$ waiting customers, anyone who arrives while the waiting room is full leaves, without coming back). Assume that customers arrive at a rate λ, and the average service time is $\frac{1}{\mu}$. Also, find the expected number of customers in the system (i.e. both waiting and being serviced). What is this expected value when $N = 8$ and $\frac{\lambda}{\mu} = 1.4$?

7. For a continuous-time Markov process with the following (per hour) rates

$$\begin{bmatrix} \times & 1.4 & 2.6 \\ 3.2 & \times & 1.9 \\ 2.0 & 3.5 & \times \end{bmatrix}$$

   (a) find the corresponding stationary distribution,

   (b) if the process is currently in State 2, what is the probability that the process will be, 25 minutes later, in State 3?

8. Consider a Brownian motion with no drift and the diffusion coefficient of $31.7\frac{\text{m}^2}{\text{day}}$. If the process starts at -13.9 m, what is the probability that, 3 days later, it will have a value between 0 and -13.9 m, *without ever* reaching 0? Hint: First, find the probability that, under the same conditions, the process will have a value less than -13.9, then less than 0.

9. Consider a Brownian motion with a drift of $-5.2\frac{\text{mm}}{\text{hr}}$ and a diffusion coefficient of $7.3\frac{\text{mm}^2}{\text{hr}}$. Evaluate:

   (a) $\Pr\{X(10{:}13) < 26\,\text{mm} \mid X(9{:}31) = 30\,\text{mm}\}$

   (b) $\Pr\{X(10{:}13) < 26\,\text{mm} \mid X(9{:}31) = 30\,\text{mm} \cap X(10{:}42) = 27\,\text{mm}\}$

10. Find the first five (up to and including ρ_5) serial correlation coefficients of the following autoregressive model

$$X_i = 1.2\, X_{i-1} - 0.7\, X_{i-2} + 0.4\, X_{i-3} + \epsilon_i$$

where $\epsilon_i \in \mathcal{N}(0,1)$. Also, what is the stationary variance of X_i, and the value of the partial correlation coefficient $\rho(X_i, X_{i-3} |\, X_{i-2})$?

11. Find all elements of the following matrix

$$\begin{bmatrix} 0 & 0.3 & 0.7 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.3 & 0.4 \\ 0.1 & 0 & 0.1 & 0.5 & 0.3 \end{bmatrix}^{10001}$$

12. Solve the following difference equation

$$5a_{i+1} - 8a_i + 5a_{i-1} = \frac{9\, i}{2^{i+1}}$$

knowing that $a_0 = \frac{35}{3}$ and $a_1 = \frac{28}{3}$. What is the value of a_{11}?

Solution:

13. $\begin{pmatrix} .2 & .3 & .5 \\ .1 & .2 & .7 \\ .6 & .2 & .2 \end{pmatrix}^2 \rightarrow \begin{matrix} .3 & .3 & .4 \end{matrix} \begin{matrix} .37 & .22 & .41 \\ .46 & .21 & .33 \\ .26 & .26 & .48 \end{matrix} \rightarrow \begin{matrix} .353 & .233 & .414 \end{matrix}$

    (a) $0.414 \times 0.26 = 0.10764$

    (b) $\begin{pmatrix} .2 & .3 & .5 \\ .1 & .2 & .7 \\ .6 & .2 & .2 \end{pmatrix}^{20} \rightarrow \begin{matrix} .344828 & .234483 & .42069 \\ .344828 & .234483 & .42069 \\ .344828 & .234483 & .42069 \end{matrix}$

    (c) $.3 + .4 \times \frac{6}{8} = .6$

14. $(.31 + .34s + .25s^2 + .1s^3)$

    (a) 13.09%

    (b) 28.78%

    (c) $4 \times 1.14^5 = 7.70166$, $4 \times 0.9404 \times \frac{1.14^4(1.14^5-1)}{.14} \rightarrow \sqrt{41.9953} = 6.48038$

15. $Q(s) = \frac{1+q^4ps^5}{q^4p^2s^6} \rightarrow \mu = \frac{6^6+5^46}{5^4} = 80.6496$, $\sigma^2 = \frac{-12}{q^4p^2} - \frac{2}{p} + \mu + \mu^2 = -\frac{12\times 6^6}{5^4} - 12 + 80.6496 + 80.6496^2 \rightarrow \sqrt{5677.21} = 75.3473$

16. _

    (a) $\Lambda = \frac{24\times 13.2}{60} = 5.28 \rightarrow 1 - e^{-5.28}(1 + 5.28 + \frac{5.28^2}{2} + \frac{5.28^3}{6} + \frac{5.28^4}{24} + \frac{5.28^5}{120}) = .433046$

    (b) $\Lambda = (2\cdot 12 - (2\cdot 12 + 30)\exp(-\frac{30}{12}))\frac{13.2}{60} = 4.30483 \rightarrow 1 - e^{-4.30483}(1 + 4.30483 + \frac{4.30483^2}{2} + \frac{4.30483^3}{6} + \frac{4.30483^4}{24} + \frac{4.30483^5}{120}) = .264141$

17. _

    (a) $\frac{13.2\times 24}{60}(1-\exp(-\frac{30}{24}))+3\exp(-\frac{30}{24}) = 4.62677$, $\frac{13.2\times 24}{60}(1-\exp(-\frac{30}{24}))+3\exp(-\frac{30}{24})(1-\exp(-\frac{30}{24})) \rightarrow \sqrt{4.38051} = 2.09297$

    (b) $\Lambda = \frac{13.2\times 24}{60} = 5.28 \rightarrow 43.30\%$ (same as before).

18. $\frac{(1-\rho)\rho^i}{1-\rho^{N+1}}$, $\rho\frac{1-\rho^{N+1}-(N+1)\rho^N(1-\rho)}{(1-\rho)(1-\rho^{N+1})}$, $1.4\frac{1-1.4^9-9\cdot 1.4^8(1-1.4)}{(1-1.4)(1-1.4^9)} = 5.9578$

19. _

    (a) 39.54%, 31.04%, 29.42%

    (b) 27.63%

20. $2\Pr(0 < Z < \frac{13.9}{\sqrt{31.7\times 3}}) - \Pr(0 < Z < \frac{2\times 13.9}{\sqrt{31.7\times 3}}) = 2\Pr(0 < Z < 1.42536) - \Pr(0 < Z < 2.85072) = \operatorname{erf}\left(\frac{1.42536}{\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{2.85072}{\sqrt{2}}\right) = 34.81\%$

21. _

(a) $\Pr(X < -4)$ where $X \in \mathcal{N}(-\frac{42}{60} \times 5.2, \sqrt{7.3 \times \frac{42}{60}}) \Rightarrow \Pr(Z < \frac{-4+3.64}{2.26053}) = \Pr(Z < -0.159255) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{0.159255}{\sqrt{2}}\right) = 43.67\%$

(b) $\Pr(X < 26)$ where $X \in \mathcal{N}(30 - \frac{3\times 42}{71}, \sqrt{7.3 \times \frac{42\times 29}{71\times 60}}) \Rightarrow \Pr(Z < \frac{26-28.2254}{1.44471}) = \Pr(Z < -1.54038) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{1.54038}{\sqrt{2}}\right) = 6.173\%$

22. $0.867925, 0.688679, 0.618868, 0.607736, 0.571547$, $V = 5.18084$, $\rho_{13,2} = 0.0587173$

23. $$\begin{bmatrix} 0 & 0.3 & 0.7 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.3 & 0.4 \\ 0.1 & 0 & 0.1 & 0.5 & 0.3 \end{bmatrix}^{10001} = \begin{bmatrix} 0 & .3 & .7 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ .45637 & .16308 & .38053 & 0 & 0 \\ .51677 & .14496 & .33825 & 0 & 0 \end{bmatrix}$$

24. $a_i = 5\left(\frac{4+3i}{5}\right)^i + 5\left(\frac{4-3i}{5}\right)^i + \frac{i+\frac{15}{9}}{2^i}$, $a_{11} = 7.0067$