BROCK UNIVERSITY

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Course: MATH 4F21 — Number of students: 2
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This is an open-book exam.
Full credit given for 7 complete answers.

1. Consider a branching process with six initial members (Generation 0), and the following probability distribution for the number of offsprings (of any of its members):

#	0	1	2	3
Pr:	0.31	0.26	0.23	0.20

   Find:

   (a) The expected value and standard deviation of the number of members of Generation 4.
   (b) The probability that this generation will have more than 17 members.
   (c) The probability of ultimate extinction.
   (d) The probability of getting extinct during the first five generations.

2. For a Markov chain with the following transition probability matrix

$$\mathbb{P} = \begin{bmatrix} 0 & 0 & 0.24 & 0.76 & 0 & 0 \\ 0 & 0 & 0.48 & 0.52 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.69 & 0.31 \\ 0 & 0 & 0 & 0 & 0.82 & 0.18 \\ 0.37 & 0.63 & 0 & 0 & 0 & 0 \\ 0.12 & 0.88 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   compute, exactly (i.e. using fractions)

   (a)
$$\lim_{n\to\infty} \mathbb{P}^{3n+1}$$
   (b) fixed probability vector,
   (c) long-run proportion of visits to State 4.

3. Solve the following difference equation

$$a_{i+1} - 3a_i + 2a_{i-1} = 2^i + 1$$

where $a_0 = 3$ and $a_{12} = 3$.

4. If S indicates getting a six when rolling a die, and F means getting any other number, find the probability generation function of:

   (a) the number of trials to generate the pattern

   $$SFFSF$$

   for the first time, and the corresponding mean and standard deviation.

   (b) the *additional* number of trials to generate the same pattern the second time (the two occurrences *are* allowed to overlap)

   (c) the *total* number of trials to generate altogether *five* occurrences of this pattern (again, consecutive occurrences may overlap).

5. For a Markov chain with the following probability transition matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.21 & 0.13 & 0.32 & 0.11 & 0.23 \\ 0.13 & 0.05 & 0.27 & 0.34 & 0.21 \\ 0.10 & 0.19 & 0.31 & 0.25 & 0.15 \end{bmatrix}$$

and the following initial distribution

State	1	2	3	4	5
Pr:	0.07	0.12	0.26	0.31	0.24

compute:

   (a) the probability of ending up in State 1,

   (b) expected value and standard deviation of the number of transitions till absorption,

   (c) expected number of visits to State 4.

6. Using the transition probability matrix of the previous question, compute:

   (a)
   $$\Pr(X_7 = 1 \mid X_5 = 3 \cap X_2 = 4)$$

   (b)
   $$\Pr(X_5 = 3 \cap X_7 = 1 \mid X_2 = 4)$$

   (c)
   $$\Pr(X_5 = 3 \mid X_7 = 1 \cap X_2 = 4)$$

7. Consider the following random experiment: Three dice are rolled, followed by flipping a coin as many times as the total number of dots on the dice. Find:

   (a) the probability generating function of the number of heads thus obtained,

   (b) the corresponding mean and standard deviation.

8. Consider a branching process with four initial members, and the distribution of the number of offsprings having the following probability generating function:

   $$F(s) = \exp\left(\frac{9s - 9}{20 - 10s}\right)$$

   Compute the mean and standard deviation of total progeny, assuming:

   (a) the process is left to run till extinction,

   (b) the process is run for five generations only (hint: using Maple, find the corresponding probability generating function first).

9. If the pattern $HTHTH$ is played against $TTHHTT$ (assuming a coin is flipped till one of these is generated), find:

   (a) the probability of $HTHTH$ winning,

   (b) the expected duration of the game,

   (c) the probability that the game will take more than 40 flips.

10. Do a complete classification of states of the following transition probability matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times \\ 0 & \times & 0 & 0 & \times & 0 & 0 & 0 \\ \times & 0 & \times & \times & 0 & \times & \times & \times \\ \times & \times & 0 & \times & \times & \times & 0 & \times \\ 0 & \times & 0 & 0 & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times \\ \times & 0 & \times & \times & 0 & \times & \times & \times \\ \times & 0 & 0 & 0 & 0 & \times & 0 & 0 \end{bmatrix}$$