BROCK UNIVERSITY

Final Examination: April 2018	Number of Pages: 3
Course: MATH 4P84	Number of students: 8
Date of Examination: April 14, 2018	Number of Hours: 3
Time of Examination: 14:00-17:00	Instructor: J. Vrbik

Open-book exam. Full credit given for 18 (out of 27) complete and correct answers; all must be entered in your booklet (as soon as computed); e-mail your Maple to jvrbik@brocku.ca (keep a copy).

1. Consider a Markov chain having the following TPM

$$\mathbb{P} = \begin{bmatrix} 0.64 & 0 & 0 & 0.36 & 0 & 0 & 0 & 0 & 0 \\ 0.16 & 0.18 & 0.12 & 0.06 & 0.04 & 0.08 & 0.15 & 0.16 & 0.05 \\ 0 & 0 & 0 & 0 & 0.75 & 0 & 0 & 0.25 & 0 \\ 0.77 & 0 & 0 & 0.23 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.23 & 0 & 0 & 0.48 & 0 & 0 & 0.29 \\ 0 & 0 & 0 & 0 & 0.45 & 0 & 0 & 0.55 & 0 \\ 0.10 & 0.14 & 0.02 & 0.15 & 0.13 & 0.19 & 0.09 & 0.06 & 0.12 \\ 0 & 0 & 0.32 & 0 & 0 & 0.55 & 0 & 0 & 0.13 \\ 0 & 0 & 0 & 0 & 0.13 & 0 & 0 & 0.87 & 0 \end{bmatrix}$$

   (a) Do a complete classification of its states (into recurrent/transient classes and subclasses). Then compute

   (b) expected number of transitions to reach a recurrent class (and the corresponding standard deviation), given the process starts in State 2,

   (c) $\Pr(X_{204} = 3 \cap X_{206} = 6 \mid X_{210} = 9 \cap X_{213} = 5)$ assuming the process is in its stationary mode (hint: pull out the relevant class, say $\mathbb{R}$, and find the corresponding $\mathring{\mathbb{R}}$).

   (d) The following four answers must be given in *fractional* form (partial credit given for decimal values)

$$\lim_{n\to\infty} (\mathbb{P}^n)_{4,1}$$

   (e)

$$\lim_{n\to\infty} (\mathbb{P}^{2n+1})_{9,5}$$

   (f)

$$\lim_{n\to\infty} (\mathbb{P}^n)_{7,4}$$

   (g)

$$\lim_{n\to\infty} (\mathbb{P}^{2n+1})_{2,6}$$

2. Consider a branching process with 13 initial members (in Generation 0), whose offspring distribution has the following PGF:

$$P(z) = \frac{1}{2 - \exp(z-1)}$$

Compute the

(a) probability that Generation 7 will have more than 25 members,

(b) expected progeny of the process up to and including Generation 7, and the corresponding standard deviation,

(c) probability that the progeny of the process up to and including Generation 7 will consist of more than 100 members,

(d) probability that extinction does not take more than 20 generations.

3. Bob an Alice bet \$3 each on the outcome of the following random experiment: 8 random cards are dealt from a standard deck of 52 cards, followed by rolling a die as many times as the number of spades in this hand; if more than 6 dots (in total) are obtained, Alice wins the round (otherwise, Bob does). (Hint: first build the PGF of the number of spades dealt, then replace its z by the PGF of the number of dots when rolling a die once; this yields the PGF of the total number of dots obtained). Compute:

(a) the probability of Alice winning a single round,

(b) the probability of her winning the game, if she starts with \$36 (Bob's initial capital is \$27), and they play till one of them goes broke,

(c) the expected number of rounds (and the corresponding standard deviation) to complete this game,

(d) the probability that the game will end in fewer than 100 rounds.

4. Consider the following difference equation

$$a_{n+2} - 2a_{n+1} + 8a_{n-1} - 12a_{n-2} + 8a_{n-3} = n^2 \cdot (-2)^{n+2}$$

(note the 'missing' a_n). Find

(a) a particular solution,

(b) a general solution,

(c) the solution which meets: $a_{-5} = -\dfrac{4543}{2000}$, $a_{-3} = \dfrac{18351}{2500}$, $a_{-1} = \dfrac{4327}{625}$, $a_0 = 4$ and $a_2 = \dfrac{5272}{625}$.

5. Consider rolling a die whose two sides are painted red, two are painted blue and two are green, until generating 3 occurrences of the pattern $RBGRG$. Find

   (a) the expected number of rolls and the corresponding standard deviation,

   (b) the probability that this will take between 300 and 500 rolls (inclusive).

   (c) Compute the expected number of occurrences of this pattern in 500 rolls of a die, and the corresponding standard deviation.

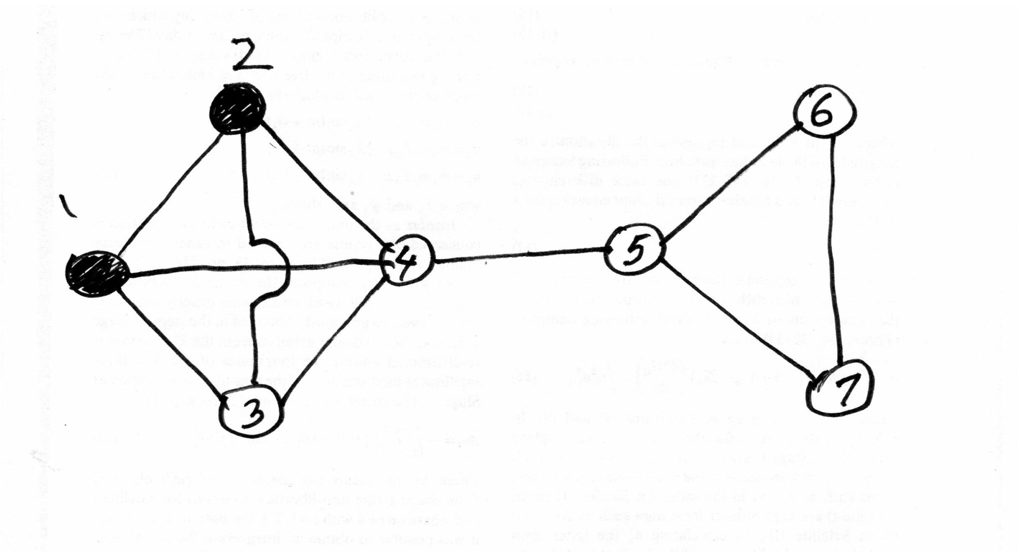
6. Consider rolling the same die until one of the following two patterns appears (for the first time): RGG or GGR. Find

   (a) the expected number of rolls this game will take, and the corresponding standard deviation,

   (b) the probability that the game will take more than 30 rolls,

   (c) the probability that RGG appears first (thus 'winning' the game).

7. Consider the following random walk



   starting in State 7. Find

   (a) the expected number of transitions till absorption in one of the two absorbing states (solid circles) and the corresponding standard deviation,

   (b) the probability of getting absorbed by State 1,

   (c) the probability of returning back to State 7 (at least once) before absorption.