BROCK UNIVERSITY

Final Examination: April 2013	Number of Pages: 3
Course: MATH 4F84	Number of students: 10
Date of Examination: Dec. 6, 2013	Number of Hours: 3
Time of Examination: 12:00-15:00	Instructor: J. Vrbik

Open book exam. Use of Maple is allowed.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Translation dictionaries (e.g. English-French) or other dictionaries (thesaurus, definitions, technical) are not allowed unless specified by the instructor and indicated on the examination paper.

Full credit given for 7 complete answers.

Numerical answers must be correct to 4 significant digits.

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1. Consider a Markov chain with the following transition probability matrix:

$$\mathbb{P} = \begin{bmatrix} 0.13 & 0.32 & 0.28 & 0.27 \\ 0.14 & 0.25 & 0.42 & 0.19 \\ 0.38 & 0.28 & 0.14 & 0.20 \\ 0.29 & 0.11 & 0.47 & 0.13 \end{bmatrix}$$

   (a) Find the corresponding stationary probability vector $\mathbf{s}$.

   (b) Assuming that $\mathbf{s}$ is used as the initial distribution (of X_0), find $\Pr(X_3 = 2 \cap X_5 = 4 \cap X_6 = 1)$,

   (c) and $\Pr(X_1 = 2 \mid X_4 = 1)$.

2. Continuation of the previous question: now, assume that $X_0 = 3$. Find the

   (a) probability that the process will visit State 4 before visiting State 1,

   (b) the expected number of transitions till the first visit to State 2,

   (c) till the second visit to State 2.

3. Consider a branching process with the offspring distribution having the following probability function

$$f(n) = 4 \cdot \frac{n+1}{3^{n+2}} \qquad n = 0, 1, 2, \ldots$$

   and having 5 initial members (in Generation 0). Compute

   (a) the probability of its extinction within the first eight generations,

   (b) of its ultimate extinction,

   (c) the expected value and standard deviation of the number of members of Generation 200,

   (d) the probability that Generation 8 has between 4 and 9 members (inclusive).

4. Find the expected value and the corresponding standard deviation of the number of (possibly overlapping) occurrences of the pattern $HHTTHH$ in 200 flips of a coin.

   What is the probability of getting exactly 5 such (possibly overlapping) occurrences?

5. Find (in terms of exact fractions)

$$\lim_{k\to\infty}\begin{bmatrix} 0 & 0.33 & 0.42 & 0 & 0.25 \\ 0.41 & 0 & 0 & 0.59 & 0 \\ 0.63 & 0 & 0 & 0.37 & 0 \\ 0 & 0.49 & 0.13 & 0 & 0.38 \\ 0.44 & 0 & 0 & 0.56 & 0 \end{bmatrix}^{2k+1}$$

6. Consider flipping a coin repeatedly till generating the first occurrence of either the $HHHH$ or the $THTHT$ pattern. What is the probability it will be the former ($HHHH$)? What is the expected number of flips needed, and the corresponding standard deviation?

7. Consider a branching process with 3 initial members (Generation 0), and the following offspring distribution:

$X=$	0	1	2	3	4
Pr	0.39	0.34	0.20	0.06	0.01

   Compute the

   (a) probability that truncated progeny up to (and including) Generation 8 will have between 10 and 30 members (inclusive),

   (b) expected number of generations till extinction, and the corresponding standard deviation.

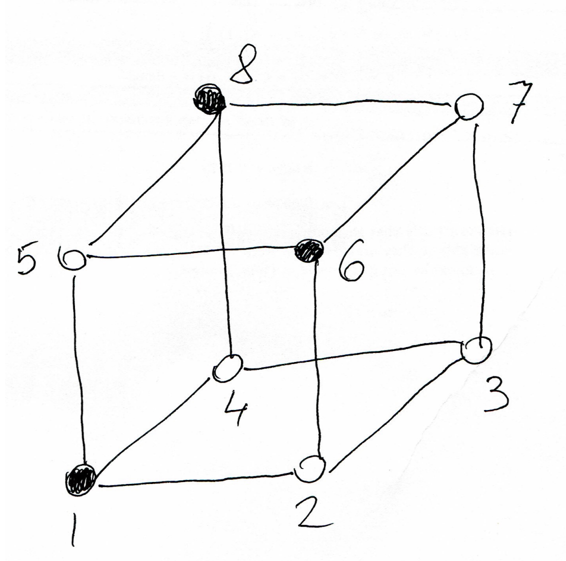
   (c) the expected value and standard deviation of total progeny.

8. Solve the following difference equation

$$3a_{n+2} - 7a_{n+1} + 5a_n - a_{n-1} = n^2$$

   knowing that $a_0 = 2$ and $a_1 = -3$ and $a_2 = 0$.

9. Consider the following random walk (solid circles are absorbing), starting randomly in any one of the five non-absorbing nodes (each has the same probability of $\frac{1}{5}$ to be chosen):



Compute the

(a) probability of ending up in State 1,

(b) expected number of moves till absorption, and the corresponding standard deviation,

(c) expected number of visits to State 5.

10. Alice and Bob play the following game: they roll a die till a second 6 is obtained (let us assume that they do not stop rolling until they do) - if this takes more than 10 rolls, Alice pays \$5 to Bob, otherwise Bob pays \$5 to Alice. Alice starts with \$75 in her pocket, Bob with \$105, and they agree to play till one of them goes broke. What is the probability of Alice winning Bob's \$105? What is the expected number of rounds of this game, and the corresponding standard deviation? What is the expected number of times the die has to be rolled, and the corresponding standard deviation?

11. Do a complete classification of a FMC with the following TPM:

$$\mathbb{P} = \begin{bmatrix} 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.3 & 0 & 0.3 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}$$

(indicate which classes are recurrent and which are transient, and what is the period of each). Also, find $\lim_{k\to\infty}(\mathbb{P}^{2k})_{4,4}$.