BROCK UNIVERSITY

Final Examination: December 2015	Number of Pages: 3
Course: MATH 4F84	Number of students: 13
Date of Examination: Dec. 21, 2015	Number of Hours: 3
Time of Examination: 16:00-19:00	Instructor: J. Vrbik

This is an open-book exam. Full credit given for 20 (out of 33) correct (to at least 4 significant digits) and complete answers. All answers **must** be entered in the examination booklet (rough work and Maple may be attached).

1. Consider a Markov chain with the following TPM:

$$\mathbb{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.2 & 0 & 0.7 \\ 0 & 0 & 0.3 & 0 & 0 & 0.7 & 0 \\ 0.2 & 0 & 0 & 0 & 0 & 0.1 & 0.7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0.2 & 0.4 & 0.1 & 0 & 0 \end{bmatrix}$$

   (a) Do a complete classification of its states.

   If the initial state is randomly selected using the following distribution (denoted $\mathbf{d}$)

$i =$	1	2	3	4	5	6	7
Pr	0.1	0.2	0.1	0	0.3	0.1	0.2

   find, in terms of *exact fractions* (only *half* credit given for decimal answers):

   (b) the expected number of transitions to reach a recurrent class, and the corresponding standard deviation,

   (c) the probability that it will take more than 5 transitions to reach a recurrent class,

   (d) the expected number of visits to State 7,

   (e)
$$\lim_{n\to\infty} \Pr(X_n = 3 \cap X_{n+2} = 6 \mid X_0 \epsilon \mathbf{d})$$

   (f)
$$\lim_{n\to\infty} \Pr(X_{2n} = 1 \mid X_0 \epsilon \mathbf{d})$$

   (g)
$$\lim_{n\to\infty} \Pr(X_n = 1 \cap X_{n+2} = 5 \mid X_0 \epsilon \mathbf{d})$$

   (h)
$$\lim_{n\to\infty} \Pr(X_n = 3 \mid X_{n+2} = 6 \cap X_0 \epsilon \mathbf{d})$$

2. Consider a branching process with 15 initial members (in Generation 0) whose offspring distribution has the following PGF

$$P(z) = \frac{0.53}{1 - 0.47z}$$

Compute:

(a) the expected number of generations till extinction and the corresponding standard deviation,

(b) the expected number of members of Generation 11 and the corresponding standard deviation,

(c) the probability that Generation 11 has more than 7 members,

(d) the expected progeny of the process up to and including Generation 11, and the corresponding standard deviation,

(e) the probability that this progeny (up to Generation 11) has more than 100 members,

(f) the probability that the *total* progeny will have more than 100 members.

Now, changing the offspring distribution to

# of offspring	0	1	2	3	4
Pr:	0.45	0.25	0.18	0.09	0.03

while keeping the initial value of 15

(g) re-do Part (d) of this question,

(h) re-do Part (e) of this question.

3. When flipping a coin, we are interested in generating (repeatedly) the following pattern of consecutive outcomes: HTHTH. What is the probability that a completion (not necessarily the first one) of this pattern is achieved on the 12^{th} flip, assuming that the individual occurrences

(a) are *not* allowed to overlap,

(b) *are* allowed to overlap.

In the second part of this question, we keep on flipping until we observe the *third* occurrence of this pattern, *allowing* the individual occurrences to overlap (e.g. the sequence HTHTHTHTH has achieved 3 occurrences already). Find

(c) the expected number of flips needed, and the corresponding standard deviation,

(d) the probability that this (generating the third occurrence) can be achieved in fewer than 100 flips.

In the last part, we flip the coin 100 times (occurrences - if any - of HTHTH are still allowed to overlap). Find

(e) the expected number of such occurrences, and the corresponding standard deviation,

(f) the probability of generating more than 5 occurrences of this pattern.

4. We roll two dice, then flip a coin as many times as the total number of dots shown. Find

   (a) the expected number of heads, and the corresponding standard deviation,

   (b) the probability of getting more than 6 heads.

5. Tom and Bob bet \$2 each on the outcome of dealing 6 cards (from a well shuffled deck of 52 cards - this has to be done repeatedly, with the 6 cards returned to the deck, in each round). Tom wins Bob's \$2 when dealing at least 2 spades, Bob wins otherwise. Tom starts with \$48, Bob with \$54, and they agree to play till one of them goes broke. What is

   (a) the probability of Tom winning the game,

   (b) the expected number of rounds they will have to play, and the corresponding standard deviation.

6. Consider the following difference equation

$$25a_{n+2} - 15a_{n+1} - 24a_n + 16a_{n-1} = 1 + (n-1)^2 \left(\frac{4}{5}\right)^{n-3}$$

   Find (without any help from 'rsolve')

   (a) the general solution to the homogeneous version of this equation (i.e. taking the RHS to be zero),

   (b) a particular solution to the whole equation,

   (c) the solution (of the whole equation) which meets the following conditions: $a_0 = 3$, $a_1 = -1$ and $a_2 = 0$.

   (d) Based on the previous solution, compute a_{10}.

7. Consider rolling a die until either $6EE6$ or $E66E$ is observed (6 stands for a six, E for any other number of dots). Find

   (a) the expected number of rolls needed, and the corresponding standard deviation,

   (b) the probability of having to roll the die more than 50 times,

   (c) the probability of $6EE6$ 'winning' (appearing before $E66E$ does).