BROCK UNIVERSITY

Final Examination: April 2019	Number of Pages: 3
Course: MATH 4P84	Number of students: 13
Date of Examination: April 12, 2019	Number of Hours: 3
Time of Examination: 19:00 - 22:00	Instructor: J. Vrbik

This is an open-book exam. Full credit given for 18 (out of 30) complete and correct answers - these must be entered in your booklet as soon as computed; e-mail your Maple to jvrbik@brocku.ca (and keep a copy).

1. Consider a Markov chain having the following TPM

$$\mathbb{P} = \begin{bmatrix} 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{10} & 0 & 0 & \frac{2}{5} & 0 & 0 & \frac{1}{10} & 0 & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{10} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{10} & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & \frac{3}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{10} & 0 & 0 & 0 & \frac{3}{10} & 0 & 0 & 0 & \frac{3}{5} \\ 0 & \frac{7}{10} & 0 & 0 & \frac{3}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{10} & \frac{3}{10} & 0 & \frac{1}{10} & 0 & 0 & 0 & \frac{3}{10} \end{bmatrix}$$

(a) Do a complete classification of its states (into recurrent/transient classes and subclasses).

Then compute

(b) the expected number of transitions (and the corresponding standard deviation) to reach a recurrent state, given that the process starts in State **6**,

(c) $\Pr(X_{208} = \mathbf{9} \cap X_{210} = \mathbf{1} \mid X_{204} = \mathbf{7} \cap X_{206} = \mathbf{4})$,

(d) $\Pr(X_{204} = \mathbf{7} \cap X_{206} = \mathbf{4} \mid X_{208} = \mathbf{9} \cap X_{210} = \mathbf{1})$ assuming that the process is in a stationary mode.

The following four answers must be computed and quoted using *fractions* only.

(e) $\lim_{n\to\infty}(\mathbb{P}^n)_{\mathbf{11},\mathbf{3}}$

(f) $\lim_{n\to\infty}(\mathbb{P}^{2n+1})_{\mathbf{5},\mathbf{9}}$

(g) $\lim_{n\to\infty}(\mathbb{P}^n)_{\mathbf{6},\mathbf{10}}$

(h) $\lim_{n\to\infty}(\mathbb{P}^{2n+1})_{\mathbf{8},\mathbf{2}}$

2. Consider a branching process with 50 initial members (in Generation 0), whose offspring distribution has the following PGF:

$$P(z) = \frac{1.03}{2.03 - z}$$

Compute

(a) the probability that none of the initial members has more then 4 children,

(b) the probability that Generation 20 will have more than 25 members,

(c) the expected progeny of the process up to and including Generation 20, and the corresponding standard deviation,

(d) the probability that the *total* progeny of the process will consist of more than 800 members (hint: use the *exact* PGF),

(e) the expected number of generations till extinction, and the corresponding standard deviation.

3. Bob an Alice bet \$**2** each on the outcome of rolling two dice; when the *total* number of dots is even, Bob wins the round, when it's odd, Alice takes the money. Compute

(a) the probability that Alice wins the game, if she starts with \$40 (Bob's initial capital is \$50), and they play till one of them goes broke,

(b) the expected number of rounds (and the corresponding standard deviation) to complete this game,

(c) the probability that the game will end in fewer than 210 rounds.

4. This time Alice bets \$1 against Bob's \$2, since her probability of winning a round is only $\frac{1}{3}$ (making it a fair game).

(a) Set up a difference equation for w_i (the probability of her winning the game, given she has currently i dollars and Bob has $N-i$ dollars) and specify the corresponding 'initial/boundary' conditions for the w_i sequence.

(b) Find her probability of winning the game if she starts with \$10 against Bob's \$12.

5. Consider the following difference equation

$$3a_{n+2} - 7a_{n+1} + 5a_n - a_{n-1} - \frac{n}{3^{n-1}} = 0$$

Find

(a) its general solution,

(b) the solution which meets
$a_{-2} = \frac{131}{4}$, $a_0 = 7$ and $a_3 = -\frac{655}{108}$.

6. Consider flipping a fair coin till getting 4 occurrences of the **HTTHT** pattern (throughout this question, consecutive occurrences *are* allowed to overlap). Compute

   (a) the expected number of flips and the corresponding standard deviation,

   (b) the probability that this will take between 100 and 200 flips (inclusive).

   (c) Compute the expected number of occurrences of this pattern in 500 flips, and the corresponding standard deviation.

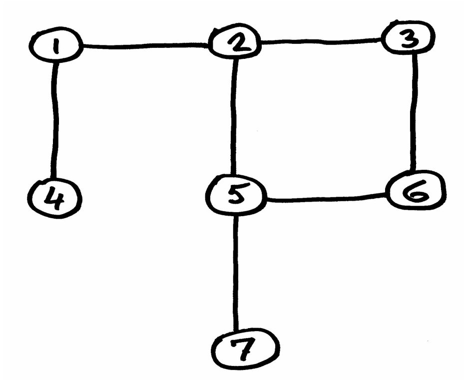
7. Consider rolling a fair die till either **6E6** or **EE6EEEEE** appears (**6** means getting six dots, **E** getting any other number) - note that these two patterns are 'incompatible'. Find

   (a) the expected number of rolls this game will take, and the corresponding standard deviation,

   (b) the probability that the game will take more than 30 rolls,

   (c) the probability that **6E6** 'wins' (appearing first).

8. Consider a random walk over the network of the next diagram



starting in a node which is selected using the following initial distribution:

$X_0 =$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Pr:	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{2}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{2}{14}$	$\frac{1}{14}$

Compute (and quote the results) using *fractions* only

   (a) the expected number of transitions to visit State **4** (for the first time), and the corresponding *variance*,

   (b) the probability that it will take more than six transitions to visit State **4** (for the first time),

   (c) the probability of visiting State **4** before either State **3** or State **7** are visited,

   (d) $$\Pr(X_9 = \mathbf{6} \mid X_{15} = \mathbf{4})$$

   Hint: recall what we know about being stationary and about being time-reversible.