MATH 4P84 SECOND MIDTERM MARCH 26, 20019

Full credit given for 6 (out of 10) correct and complete answers.

All answers (supported by Maple) must be entered in your booklet.

Open-book exam. Duration: 1 hour

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1. Consider a FMC with the following TPM and running in its stationary mode

$$\mathbb{P} = \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0 & 0.2 & 0 & 0.2 & 0 & 0.4 \\ 0.2 & 0 & 0.2 & 0.5 & 0.1 & 0 & 0 \\ 0.3 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0.2 & 0.3 & 0.2 \\ 0 & 0.2 & 0.1 & 0 & 0.3 & 0.1 & 0.3 \\ 0 & 0.1 & 0 & 0.1 & 0.4 & 0.2 & 0.2 \end{bmatrix}$$

   (a) Is it 'lumpable' in the following sense: $3, 6 \mid 2, 4, 7 \mid 1, 5$?
   If NO, give at least one reason why it isn't, if YES, construct the correspondingly reduced $\overline{\mathbb{P}}$.

   (b) Find

$$\Pr\left(X_{647} = 5 \cap X_{650} = 5 \mid X_{653} = 3 \cap X_{652} = 4\right)$$

2. Consider two players betting \$4 each on a roll of 4 dice; if at least one six (a side showing 6 dots) appears, Mr A wins the round, otherwise the money goes to Mr B. They start with \$80 (Mr A) and \$100 (Mr B) and agree to play till one of them goes broke. What is the probability that

   (a) the game will take more than 150 rounds,

   (b) Mr B ends up winning it.

3. *Without* Maple, find the general solution to the following difference equation

$$a_n - a_{n-1} - 12a_{n-2} = n \cdot 2^{2n-3}$$

4. Consider the following TPM

$$\mathbb{P} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{29}{50} & \frac{21}{50} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{47}{100} & \frac{31}{100} & \frac{11}{50} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{25} & \frac{9}{20} & \frac{19}{100} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14}{25} & \frac{11}{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{17}{50} & \frac{33}{50} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{25} & \frac{4}{25} & \frac{14}{25} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{9}{50} & 0 & 0 & 0 & \frac{11}{100} & 0 & 0 & \frac{19}{100} & 0 & \frac{13}{25} \\ 0 & 0 & \frac{13}{100} & 0 & \frac{17}{100} & 0 & \frac{7}{50} & 0 & 0 & \frac{14}{25} & 0 \end{bmatrix}$$

Using fractions only, find

(a)
$$\lim_{k\to\infty} \left(\mathbb{P}^{2k+1}\right)_{7,6}$$

(b)
$$\lim_{k\to\infty} \left(\mathbb{P}^{k}\right)_{10,1}$$

(c)
$$\lim_{k\to\infty} \left(\mathbb{P}^{2k}\right)_{11,7}$$

5. Consider a branching process whose offspring distribution is Binomial (with $n = 6$ and $p = \frac{1}{6}$) and which starts with 5 initial members. Compute (in decimal)

   (a) the probability that Generation 63 will have between 1 and 30 members (inclusive),

   (b) the expected number of its members who will have ever lived up to and including Generation 63, and the corresponding standard deviation.