Solving PDE of the type

$$\dot{P}(z,t) = a(z) \cdot P'(z,t)$$

where $a(z)$ is a specific (given) function of z.

First we note that, if $P(z,t)$ is a solution, then any function of $P(z,t)$, say $g[P(z,t)]$ is also a solution. This is clear from the chain rule:

$$\begin{aligned} \dot{g}[P(z,t)] &= \acute{g}[P(z,t)] \cdot \dot{P}(z,t) \\ g'[P(z,t)] &= \acute{g}[P(z,t)] \cdot P'(z,t) \end{aligned}$$

Substituted back into the original equation, $\acute{g}[P(z,t)]$ cancels out.

We will assume that

$$P(z,t) = Q(z) \cdot R(t)$$

Substitute and get:

$$Q(z) \cdot \dot{R}(t) = a(z) \cdot Q'(z) \cdot R(t)$$

Divide by $Q(z) \cdot R(t)$:

$$\frac{\dot{R}(t)}{R(t)} = a(z)\frac{Q'(z)}{Q(z)}$$

A function of t (no z) can be equal to a function of z (no t) only if both of them are equal to the same constant, say $-\gamma$. We thus get

$$\frac{\dot{R}(t)}{R(t)} = -\gamma$$

and

$$a(z)\frac{Q'(z)}{Q(z)} = -\gamma$$

The first of these has the following general solution

$$R(t) = c \cdot e^{-\gamma t}$$

the second one implies

$$\ln Q(z) = -\gamma \int \tfrac{dz}{a(z)}$$

or

$$Q(z) = \exp\left(-\gamma \int \tfrac{dz}{a(z)}\right)$$

We then know that

$$g\left[c \cdot e^{-\gamma t} \cdot \exp\left(-\gamma \int \tfrac{dz}{a(z)}\right)\right]$$

is also a solution, where g is any univariate function. Clearly, both the multiplication by c and raising to the power of γ can be 'absorbed' into g, so re-writing the solution as

$$g\left[e^{-t} \cdot \exp\left(-\int \tfrac{dz}{a(z)}\right)\right]$$

is basically saying the same thing. Furthermore, one can show that this is the general solution of the original equation (i.e. all its solutions have this form).

The initial condition

$$P(z,0) = z^i = g\left[\exp\left(-\int \tfrac{dz}{a(z)}\right)\right]$$

(where i is the value of the process at time $t=0$) then determines the exact form of $g(..)$.

EXAMPLES

1. Yule process, $a(z) = -\lambda z(1-z)$. Since

$$\int \tfrac{dz}{z(1-z)} = \int \left(\frac{1}{z} + \tfrac{1}{1-z}\right) dz = \ln z - \ln(1-z)$$

   we get

$$P(z,t) = g_\text{o}\left[e^{-t}\cdot \exp(\tfrac{1}{\lambda}\ln\tfrac{z}{1-z})\right] =$$
$$g_\text{o}\left[e^{-t}\cdot (\tfrac{z}{1-z})^{1/\lambda}\right] \equiv g\left[e^{-\lambda t}\cdot \tfrac{z}{1-z}\right]$$

   where $g(..)$ is such that

$$g(\tfrac{z}{1-z}) = z^i$$

   or

$$g(x) = (\tfrac{x}{1+x})^i$$

   The final solution is thus

$$P(z,t) = \left[\frac{e^{-\lambda t}\cdot\frac{z}{1-z}}{1+e^{-\lambda t}\cdot\frac{z}{1-z}}\right]^i = \left[\frac{e^{-\lambda t}\cdot z}{1-z+e^{-\lambda t}\cdot z}\right]^i$$

   This is clearly a PGF of the negative binomial distribution (number of trials to get the i^{th} success) with $p_t = e^{-\lambda t}$.

2. Pure-Death process, $a(z) = \mu(1-z)$.

$$P(z,t) = g_\text{o}\left\{e^{-t}\cdot \exp[\tfrac{1}{\mu}\ln(1-z)]\right\} =$$
$$g_\text{o}\left[e^{-t}\cdot (1-z)^{1/\mu}\right] \equiv g\left[e^{-\mu t}\cdot (1-z)\right]$$

   where $g(..)$ is such that

$$g(1-z) = z^i$$

   or

$$g(x) = (1-x)^i$$

   The final solution is thus

$$P(z,t) = [1 - e^{-\mu t}\cdot(1-z)]^i = (1 - e^{-\mu t} + e^{-\mu t}\cdot z)^i$$

   This is clearly a PGF of the binomial distribution, with $p_t = e^{-\mu t}$ and total number of trials equal to i.

3. Linear-Growth process: $a(z) = (1-z)(\mu - \lambda z)$. Since

$$\int \frac{dz}{(1-z)(\mu-\lambda z)} = \frac{1}{\mu-\lambda} \int \left(\frac{1}{1-z} - \frac{\lambda}{\mu-\lambda z}\right) dz =$$
$$\frac{1}{\mu-\lambda} \left[\ln(\mu - \lambda z) - \ln(1-z)\right]$$

we get

$$P(z,t) = g_0 \left[e^{-t} \cdot \left(\frac{1-z}{\mu-\lambda z}\right)^{\frac{1}{\mu-\lambda}}\right] =$$
$$g\left(e^{-t(\mu-\lambda)} \cdot \frac{1-z}{\mu-\lambda z}\right)$$

where $g(..)$ is such that

$$g\left(\frac{1-z}{\mu-\lambda z}\right) = z^i$$

or

$$g(x) = \left(\frac{1-\mu x}{1-\lambda x}\right)^i$$

The final solution is thus

$$P(z,t) = \left[\frac{1 - \mu \cdot e^{-t(\mu-\lambda)} \cdot \frac{1-z}{\mu-\lambda z}}{1 - \lambda \cdot e^{-t(\mu-\lambda)} \cdot \frac{1-z}{\mu-\lambda z}}\right]^i =$$
$$\left[\frac{\mu - \lambda z - \mu e^{-t(\mu-\lambda)}(1-z)}{\mu - \lambda z - \lambda e^{-t(\mu-\lambda)}(1-z)}\right]^i =$$
$$\left[\frac{\mu(1 - e^{-t(\mu-\lambda)}) - (\lambda - \mu e^{-t(\mu-\lambda)})z}{\mu - \lambda e^{-t(\mu-\lambda)} - \lambda(1 - e^{-t(\mu-\lambda)})z}\right]^i =$$
$$\left[\frac{\frac{\mu(1-e^{-t(\mu-\lambda)})}{\mu-\lambda e^{-t(\mu-\lambda)}} - \frac{\lambda-\mu e^{-t(\mu-\lambda)}}{\mu-\lambda e^{-t(\mu-\lambda)}} z}{1 - \frac{\lambda(1-e^{-t(\mu-\lambda)})}{\mu-\lambda e^{-t(\mu-\lambda)}} z}\right]^i =$$
$$\left[\frac{r_t - (r_t + q_t - 1)z}{1 - q_t z}\right]^i =$$
$$\left[\frac{r_t - r_t(q_t + p_t)z + p_t z}{1 - q_t z}\right]^i = \left[r_t + (1 - r_t)\frac{p_t z}{1 - q_t z}\right]^i$$

which is a `composition` of a Binomial distribution with i trials and $1 - r_t$ for the probability of a success, and a Geometric distribution for which the probability of a success is p_t.

Extension

to solve a slightly more complicated

$$\dot{P}(z,t) = a(z) \cdot P'(z,t) + b(z) \cdot P(z,t)$$

One can show that the general solution can be found by solving the 'homogenous' version of this equation first (which we already know how to do - I'll now call it

$G(z,t)$), and then multiplying it by a function of z, say $h(z)$, substituting into the full (non-homogeneous) equation, and solving for $h(z)$. This yields

$$\begin{aligned}&\dot{G}(z,t)h(z) = \\ &a(z)G'(z,t)h(z) + a(z)G(z,t)h'(z) + b(z)G(z,t), h(z)\end{aligned}$$

Since

$$\dot{G}(z,t) = a(z)G'(z,t)$$

we can cancel the first two terms and write

$$0 = a(z)G(z,t)h'(z) + b(z)G(z,t), h(z)$$

or

$$\frac{h'(z)}{h(z)} = -\frac{b(z)}{a(z)}$$

which can be easily solved:

$$h(z) = \exp\left(-\int \tfrac{b(z)}{a(z)} dz\right)$$

To solve the initial-value problem, we now have to find $g(..)$ such that

$$g\left[\exp\left(-\int \frac{dz}{a(z)}\right)\right] \cdot h(z) = z^i$$

`Examples:`

1. Linear Growth with immigration:

$$\begin{aligned}a(z) &= (1-z)(\mu - \lambda z)\\ b(z) &= -a(1-z)\end{aligned}$$

We have already solved the homogeneous version, so let us find

$$\begin{aligned}&h(z) = \exp\left(a \int \tfrac{1-z}{(1-z)(\mu-\lambda z)} dz\right) = \\ &\exp\left(-\frac{a}{\lambda}\ln(\mu - \lambda z)\right) = (\mu - \lambda z)^{-a/\lambda}\end{aligned}$$

The general solution is thus

$$g\left(e^{-t(\mu-\lambda)} \cdot \tfrac{1-z}{\mu-\lambda z}\right)(\mu - \lambda z)^{-a/\lambda}$$

We will first solve it assuming that $i = 0$, which implies

$$\begin{aligned}&g\left(\tfrac{1-z}{\mu-\lambda z}\right)(\mu - \lambda z)^{-a/\lambda} = 1\\ &g\left(\tfrac{1-z}{\mu-\lambda z}\right) = (\mu - \lambda z)^{a/\lambda}\\ &g(x) = (\mu - \lambda \tfrac{1-\mu x}{1-\lambda x})^{a/\lambda} = \left(\tfrac{\mu-\lambda}{1-\lambda x}\right)^{a/\lambda}\end{aligned}$$

This yields

$$\left(\frac{\mu-\lambda}{1-\lambda e^{-t(\mu-\lambda)}\cdot\frac{1-z}{\mu-\lambda z}}\right)^{a/\lambda}(\mu-\lambda z)^{-a/\lambda}=$$
$$\left(\frac{\mu-\lambda}{\mu-\lambda z-\lambda e^{-t(\mu-\lambda)}(1-z)}\right)^{a/\lambda}=$$
$$\left(\frac{\mu-\lambda}{\mu-\lambda e^{-t(\mu-\lambda)}-\lambda(1-e^{-t(\mu-\lambda)})\ z}\right)^{a/\lambda}=$$
$$\left(\frac{\frac{\mu-\lambda}{\mu-\lambda e^{-t(\mu-\lambda)}}}{1-\frac{\lambda(1-e^{-t(\mu-\lambda)})}{\mu-\lambda e^{-t(\mu-\lambda)}}\ z}\right)^{a/\lambda}=\left(\frac{p_t}{1-q_t\ z}\right)^{a/\lambda}$$

which is the *modified* Negative binomial distribution with parameters p_t and $\frac{a}{\lambda}$.

2. M/M/∞ queue:

$$\begin{aligned} a(z) &= \mu(1-z) \\ b(z) &= -a(1-z) \end{aligned}$$

The general solution to the homogeneous version of the equation is

$$g_\circ\left[e^{-t}\cdot\exp\left(-\frac{1}{\mu}\int\frac{dz}{1-z}\right)\right]=g_\circ\left[e^{-t}\cdot(1-z)^{1/\mu}\right]=$$
$$g\left[e^{-\mu t}\cdot(1-z)\right]$$

Then, we get

$$h(z)=\exp\left(\frac{a}{\mu}\int dz\right)=\exp\left(\frac{a}{\mu}z\right)$$

The general solution is thus

$$g\left[e^{-\mu t}\cdot(1-z)\right]\cdot\exp\left(\frac{a}{\mu}z\right)$$

To meet the initial condition, we need

$$g(1-z)\cdot\exp\left(\frac{a}{\mu}z\right)=z^i$$

or

$$g(1-z)=\exp\left(-\frac{a}{\mu}z\right)\cdot z^i$$

implying

$$g(x)=\exp\left[-\frac{a}{\mu}(1-x)\right]\cdot(1-x)^i$$

Finally, we have to replace x by the original argument of g, and multiply by $h(z)$:

$$\exp\left\{-\frac{a}{\mu}[1-e^{-\mu t}\cdot(1-z)]\right\}\cdot[1-e^{-\mu t}\cdot(1-z)]^i\cdot\exp\left(\frac{a}{\mu}z\right)=$$
$$\exp\left\{-\frac{a}{\mu}(1-e^{-\mu t}+e^{-\mu t}z-z)\right\}\cdot(q_t+p_t z)^i=$$

$$\exp\left[-\frac{a\cdot q_t}{\mu}(1-z)\right]\cdot(q_t+p_t z)^i$$

where $p_t = e^{-\mu t}$. This is an independent sum (convolution) of a Poisson distribution with $\Lambda = \frac{a\cdot q_t}{\mu}$, and a Binomial distribution with i trials and p_t as the probability of a success.

3. Power-supply process (N welders):

$$\begin{aligned} a(z) &= (1-z)(\mu+\lambda\, z) \\ b(z) &= -N\,\lambda\,(1-z) \end{aligned}$$

Since

$$\int \frac{dz}{(1-z)(\mu+\lambda z)} = \frac{1}{\mu+\lambda}\int\left(\frac{1}{1-z}+\frac{\lambda}{\mu+\lambda z}\right)dz =$$
$$\frac{1}{\lambda+\mu}\left[\ln(\mu+\lambda z)-\ln(1-z)\right]$$

the general solution to the homogeneous version of the equation is

$$g_{\circ}\left[e^{-t}\cdot\left(\frac{1-z}{\mu+\lambda z}\right)^{\frac{1}{\lambda+\mu}}\right] = g\left[e^{-t(\lambda+\mu)}\cdot\frac{1-z}{\mu+\lambda z}\right]$$

Then, we get

$$h(z) = \exp\left(N\lambda\int\frac{dz}{\mu+\lambda\, z}\right) = (\mu+\lambda\, z)^N$$

The general solution is thus

$$g\left[e^{-t(\lambda+\mu)}\cdot\frac{1-z}{\mu+\lambda z}\right]\cdot(\mu+\lambda\, z)^N$$

To meet the initial condition, we need

$$g\left(\frac{1-z}{\mu+\lambda z}\right)\cdot(\mu+\lambda\, z)^N = z^i$$

or

$$g\left(\frac{1-z}{\mu+\lambda z}\right) = z^i\cdot(\mu+\lambda\, z)^{-N}$$

implying

$$g(x) = \left(\frac{1-\mu x}{1+\lambda x}\right)^i\cdot\left(\frac{\lambda+\mu}{1+\lambda x}\right)^{-N}$$

Finally, we have to replace x by the original argument of g, and multiply by $h(z)$:

$$\left(\frac{1-\mu e^{-t(\lambda+\mu)}\frac{1-z}{\mu+\lambda z}}{1+\lambda e^{-t(\lambda+\mu)}\frac{1-z}{\mu+\lambda z}}\right)^i \left(\frac{1+\lambda e^{-t(\lambda+\mu)}\frac{1-z}{\mu+\lambda z}}{\lambda+\mu}\right)^N (\mu+\lambda\, z)^N =$$
$$\left(\frac{\mu+\lambda z-\mu e^{-t(\lambda+\mu)}(1-z)}{\mu+\lambda z+\lambda e^{-t(\lambda+\mu)}(1-z)}\right)^i \cdot$$
$$\left(\frac{\mu+\lambda z+\lambda e^{-t(\lambda+\mu)}(1-z)}{\lambda+\mu}\right)^N =$$
$$\left(\frac{\mu(1-e^{-t(\mu+\lambda)})+(\lambda+\mu e^{-t(\mu+\lambda)})z}{\lambda+\mu}\right)^i \cdot$$
$$\left(\frac{\mu+\lambda e^{-t(\mu+\lambda)}+\lambda(1-e^{-t(\mu+\lambda)})z}{\lambda+\mu}\right)^{N-i} =$$
$$\left(q_t^{(1)}+p_t^{(1)}z\right)^i \cdot \left(q_t^{(2)}+p_t^{(2)}z\right)^{N-i}$$

(a convolution of two Binomials), where

$$p_t^{(1)} \equiv \frac{\lambda+\mu e^{-t(\mu+\lambda)}}{\lambda+\mu}$$
$$p_t^{(2)} \equiv \frac{\lambda(1-e^{-t(\mu+\lambda)})}{\lambda+\mu}$$