POISSON PROCESS (PP): State space consists of all non-negative integers, time is continuous.

The process is time homogeneous, with independent increments and

$$\begin{aligned} P_0(t) &= 1 - \lambda \cdot t + o(t) \\ P_1(t) &= \lambda \cdot t + o(t) \\ P_2(t) &= o(t) \\ &\vdots \end{aligned}$$

With the help of the generating function $P(z,t)$ of the $P_0(t)$, $P_1(t)$, ... sequence, we have derived:

$$P_n(t) = \frac{\lambda^n t^n}{n!} e^{-\lambda t}$$

The correlation between $N(t)$ and $N(t+s)$ is

$$\frac{1}{\sqrt{1+\frac{s}{t}}}$$

The conditional distribution of $N(s)$, given that $N(t) = n$ (and $s < t$) is $\mathcal{B}(n, p = \frac{s}{t})$.

To simulate the process (create one random realization of it), from time 0 to time T, we can either

1. keep on adding independent, exponentially distributed (with mean $\frac{1}{\lambda}$) inter-arrival times, or

2. draw one random integer from a Poisson distribution with the mean of $\Lambda \equiv \lambda \cdot T$, then generate a RIS from $\mathcal{U}(0,T)$, and take the arrival time to be the corresponding order statistics (i.e. 'sorting' the sample).

Non-homogenous version of PP allows λ to be (a specific, given) function of t (often stepwise). The probability of (exactly) n arrivals during time interval (t_1, t_2) is then computed by

$$\frac{\Lambda^n}{n!} e^{-\Lambda}$$

where

$$\Lambda = \int_{t_1}^{t_2} \lambda(t)\ dt$$

We can **split** a (homogeneous) PP into two *independent* processes by 'flipping a coin' to decide whether the current arrival goes 'left' or 'right'. The corresponding λ as split accordingly ($\lambda_{\text{left}} = p \cdot \lambda$, $\lambda_{\text{right}} = q \cdot \lambda$).

Two (homogeneous) **PPs 'competing'**. Probability that the first will reach the value of n before the second one reaches m is

$$p^n \sum_{i=0}^{m-1} \binom{n-1+i}{i} q^i$$

PP in 2 or more **dimensions**. The number of 'dandelions' in an area of size A has the Poisson distribution with $\Lambda = A \cdot \lambda$, where λ is the average density.

M/G/∞ queue. $X(t)$ [customers being served] and $Y(t)$ [customers who have already left] are *independent*, Poisson-type RVs, with $\Lambda_{\mathrm{x}} = \lambda \cdot t \cdot p_t$ and $\Lambda_{\mathrm{y}} = \lambda \cdot t \cdot q_t$, where

$$p_t = \frac{1}{t}\int_0^t [1 - G(u)]\; du$$

$G(u)$ being the distribution function of the service times. Note that $\lim_{t\to\infty} \Lambda_{\mathrm{x}} = \lambda \cdot \mu_{\mathrm{sev.}}$

Compound (cluster) PP. Its MGF is

$$\exp\{-\lambda t[1 - M(u)]\}$$

M being the MGF of the 'size' distribution. For integer-type 'size' distribution ('cluster' precess), $M(u)$ can be replaced by the corresponding PGF $P(z)$, getting the PGF of $Y(t)$. In either case

$$\begin{aligned}
\mathbb{E}\left[Y(t)\right] &= \lambda t \mu_{\mathrm{size}} \\
\mathrm{Var}\left[Y(t)\right] &= \lambda t(\sigma^2_{\mathrm{size}} + \mu^2_{\mathrm{size}})
\end{aligned}$$

PP of random duration. The PGF of $N(T)$ is

$$M[\lambda(z-1)]$$

where M is the MGF of T. This implies

$$\begin{aligned}
\mathbb{E}\left[N(T)\right] &= \lambda \mu_{\mathrm{T}} \\
\mathrm{Var}\left[N(T)\right] &= \lambda \mu_{\mathrm{T}} + \lambda^2 \sigma^2_{\mathrm{T}}
\end{aligned}$$

PURE BIRTH PROCESS (PB): Only one axiom changes, namely

$$P_{n,n+1}(t) = \lambda_n \cdot t + o(t)$$

A special case is the **Yule process** with

$$\lambda_n \equiv n \cdot \lambda$$

We define the corresponding PGF

$$P_i(z,t) \equiv \sum_{n=i}^{\infty} P_{i,n}(t)\; z^n$$

which must meet the following PDEλ

$$\overset{\bullet}{P}_i(z,t) = \lambda\, z(z-1) P_i'(z,t)$$

subject to

$$P_i(z,0) = z^i$$

The solution:

$$P(z,t) = \left(\frac{z\,p_t}{1 - z\,q_t}\right)^i$$

(negative binomial), where

$$p_t = e^{-\lambda t}$$

PURE DEATH PROCESS *decreases* (rather than increases), one unit at a time, at a rate denoted μ_n.

Special case:

$$\mu_n \equiv n \cdot \mu$$

leads to

$$\overset{\bullet}{P}_i(z,t) = \mu\,(1-z)P_i'(z,t)$$

and

$$P_i(z,t) = (q_t + p_t\,z)^i$$

where $p_t = e^{-\mu t}$.

Time till extinction has the following distribution function

$$\Pr(T \le t) = (1 - e^{-\mu\,t})^i$$

the expected value

$$\frac{1}{\mu}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + ... + \tfrac{1}{i}\right)$$

and variance

$$\frac{1}{\mu^2}\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + ... + \tfrac{1}{i^2}\right)$$

BIRTH AND DEATH PROCESSES can both increase (at the rate of λ_n) or decrease (at the rate of μ_n) one unit at a time. We will consider several special cases.

Linear Growth:

$$\begin{aligned}\lambda_n &= n \cdot \lambda \\ \mu_n &= n \cdot \mu\end{aligned}$$

leading to

$$\overset{\bullet}{P}_i(z,t) = (z-1)(\lambda z - \mu)P_i'(z,t)$$

and

$$P_i(z,t) = \left(r_t + (1 - r_t)\frac{p_t\,z}{1 - q_t\,z}\right)^i$$

(a composition of Binomial and Geometric distributions), where

$$\begin{aligned}p_t &\equiv \frac{(\lambda - \mu)\,e^{-(\lambda-\mu)\,t}}{\lambda - \mu\,e^{-(\lambda-\mu)\,t}} \\ r_t &\equiv \frac{\mu(1 - e^{-(\lambda-\mu)\,t})}{\lambda - \mu\,e^{-(\lambda-\mu)\,t}}\end{aligned}$$

The corresponding expected value and variance:

$$
\begin{aligned}
&i \cdot e^{(\lambda-\mu)\,t} \\
&i \cdot \frac{\lambda+\mu}{\lambda-\mu}(e^{(\lambda-\mu)\,t}-1)\,e^{(\lambda-\mu)\,t}
\end{aligned}
$$

Probability of extinction (already) at time t is r_t^i, probability of *ultimate* extinction is r_∞^i (certain when $\lambda \le \mu$).

Expected time till extinction (meaningful only when this is certain):

$$
\omega_1 \equiv \frac{1}{\lambda} \ln \frac{\mu}{\mu-\lambda}
$$

when $i = 1$, and generated by

$$
\omega_{i+1} = \left(1+\tfrac{\mu}{\lambda}\right)\omega_i - \tfrac{\mu}{\lambda}\,\omega_{i-1} - \tfrac{1}{i\,\lambda}
$$

when $i > 1$ (utilizing $\omega_0 = 0$).

Linear Growth with Immigration:

$$
\begin{aligned}
\lambda_n &= n \cdot \lambda + a \\
\mu_n &= n \cdot \mu
\end{aligned}
$$

leading to

$$
\overset{\bullet}{P}_i(z,t) = (z-1)(\lambda\,z-\mu)P_i'(z,t) + a(z-1)P_i(z,t)
$$

(non-homogeneous), having the following $i = 0$ solution:

$$
P_0(z,t) = \left(\frac{p_t}{1-z\,q_t}\right)^{a/\lambda}
$$

(modified negative binomial), with the mean of $\dfrac{aq_t}{\lambda p_t}$ and variance $\dfrac{aq_t}{\lambda p_t^2}$.

The general (any i) solution is:

$$
P_i(z,t) = \left(\frac{p_t}{1-z\,q_t}\right)^{a/\lambda} \cdot \left(r_t + (1-r_t)\frac{p_t\,z}{1-q_t\,z}\right)^i
$$

The stationary distribution (only when $\lambda < \mu$) is

$$
P_i(z,\infty) = \left(\frac{\mu-\lambda}{\mu-\lambda\,z}\right)^{a/\lambda}
$$

M/M/∞ queue:

$$
\begin{aligned}
\lambda_n &= a \\
\mu_n &= n \cdot \mu
\end{aligned}
$$

leads to

$$\overset{\bullet}{P}_i(z,t) = \mu\,(1-z)P_i'(z,t) + a(z-1)P_i(z,t)$$

Its solution is

$$P_i(z,t) = \exp\left(\frac{a\,q_t}{\mu}\cdot(z-1)\right)\cdot(q_t + p_t\,z)^i$$

where now $p_t = e^{-\mu t}$. This is a convolutions (i.e. independent sum) of Poisson (with the mean of $\frac{a\,q_t}{\mu}$) and binomial. Clearly, the asymptotic distribution is

$$P_i(z,\infty) = \exp\left(\frac{a}{\mu}\cdot(z-1)\right)$$

N welders:

$$\begin{aligned}\lambda_i &= \lambda\cdot(N-n)\\ \mu_i &= \mu\cdot n\end{aligned}$$

leading to

$$\overset{\bullet}{P}_i(z,t) = (1-z)(\mu+\lambda\,z)P_i'(z,t) + N\,\lambda\,(z-1)\,P_i(z,t)$$

Solution:

$$P_i(z,t) = (q_t^{(1)} + p_t^{(1)}z)^i\cdot(q_t^{(2)} + p_t^{(2)}z)^{N-i}$$

(a convolution of two binomials), where

$$\begin{aligned}p_t^{(1)} &= \frac{\lambda + \mu\,e^{-(\lambda+\mu)\,t}}{\lambda+\mu}\\ p_t^{(2)} &= \frac{\lambda(1-e^{-(\lambda+\mu)\,t})}{\lambda+\mu}\end{aligned}$$

Clearly

$$P_i(z,\infty) = \left(\frac{\mu+\lambda\;z}{\mu+\lambda}\right)^N$$

Other (more complicated) models:

We give up on $P_i(z,t)$, and try to get the corresponding **stationary distribution** ($t\to\infty$) only, by:

$$p_n = \frac{\lambda_0\lambda_1\lambda_2...\lambda_{n-1}}{\mu_1\mu_2\mu_3...\mu_n}\,p_0$$

where

$$p_0 = \left(\sum_{n=0}^{\infty}\frac{\lambda_0\lambda_1\lambda_2...\lambda_{n-1}}{\mu_1\mu_2\mu_3...\mu_n}\right)^{-1}$$

If the sum is infinite (diverges), stationary distribution does not exist.

Examples:
`M/M/1 queue:`

$$p_n = \rho^n(1-\rho)$$

where $\rho \equiv \frac{\lambda}{\mu}$.

Average number of people in the system: $\frac{\rho}{1-\rho}$, people waiting: $\frac{\rho^2}{1-\rho}$. SUF: ρ. Average length of idle period: $\frac{1}{\lambda}$, busy period: $\frac{1}{\mu-\lambda}$, busy cycle: the sum of the two.

`M/M/c queue:`

$$p_n = \begin{cases} \dfrac{\rho^n}{n!\Gamma} & n \le c \\ \dfrac{\rho^n}{c!\Gamma c^{n-c}} & n \ge c \end{cases}$$

where

$$\Gamma = \sum_{i=0}^{c-1} \frac{\rho^i}{i!} + \frac{\rho^c}{c!(1-\frac{\rho}{c})}$$

Average number of busy servers is ρ, average length of the actual queue is

$$\frac{\rho^c}{c!\Gamma} \cdot \frac{\frac{\rho}{c}}{(1-\frac{\rho}{c})^2}$$

The expected length of full-utilization period is $(c \cdot \mu - \lambda)^{-1}$. ■

When State 0 is absorbing ($\lambda_0 = 0$), stationary mode is impossible, and the main issue is the **probability of ultimate absorption**, given the process starts in State m (denoted a_m). This is computed by

$$\begin{aligned} d_n &= \frac{\prod_{k=1}^{n} \frac{\mu_k}{\lambda_k}}{\text{Sum}} \\ a_m &= 1 - \sum_{n=0}^{m-1} d_n \end{aligned}$$

(certain when the Sum diverges).

When absorption is *certain*, the **expected time** till it happens is computed, recursively, by

$$\begin{aligned} \omega_0 &= 0 \\ \omega_1 &= \sum_{i=1}^{\infty} \frac{\lambda_1\lambda_2...\lambda_{i-1}}{\mu_1\mu_2...\mu_{i-1}\mu_i} \\ \omega_{n+1} &= (1+\frac{\mu_n}{\lambda_n})\,\omega_n - \frac{\mu_n}{\lambda_n}\,\omega_{n-1} - \frac{1}{\lambda_n} \end{aligned}$$

(may turn out to be infinite - not typical though).

GENERAL TIME-CONTINUOUS MARKOV PROCESSES

can move, instantaneously, from any state to any other state. For simplicity, we consider only finitely many (usually a handful) of states. The corresponding rates can be organized in a matrix form (leaving the main diagonal elements blank). When the main diagonal elements are defined to be the negative sum of the remaining row elements, the resulting matrix $\mathbb{A}$ is called the infinitesimal generator of the process. The probability of being in State j at time t, given that at time 0 the process starts in State i, is given by the $(i,j)^{th}$ element of $\mathbb{P}(t)$, where $\mathbb{P}(t)$ is a solution to

$$\overset{\bullet}{\mathbb{P}}(t) = \mathbb{A}\,\mathbb{P}(t)$$

(Kolmogorov's forward equations), namely

$$\mathbb{P}(t) = \exp(\mathbb{A}t)$$

This requires computing a function of a square matrix, say $\mathbb{B}$, which is done in general by

$$f(\mathbb{B}) = \mathbb{C}_1 f(\omega_1) + \mathbb{C}_2 f(\omega_2) + ... + \mathbb{C}_N\, f(\omega_N)$$

where ω_k $(k = 1..N)$ are the matrix's eigenvalues, and $\mathbb{C}_k$ are the corresponding constituent matrices (N is the matrix's size). They can be computed based on

$$\begin{aligned} \sum_{k=1}^{N} \mathbb{C}_k &= \mathbb{I} \\ \sum_{k=1}^{N} \omega_k \mathbb{C}_k &= \mathbb{A} \\ \sum_{k=1}^{N} \omega_k^2 \mathbb{C}_k &= \mathbb{A}^2 \\ &\vdots \\ \sum_{k=1}^{N} \omega_k^{N-1} \mathbb{C}_k &= \mathbb{A}^{N-1} \end{aligned}$$

When two eigenvalues (say $\omega_1 = \omega_2$) are identical, $\mathbb{C}_1 f(\omega_1) + \mathbb{C}_2 f(\omega_2) + ...$ changes to $\mathbb{C}_1 f(\omega_1) + \mathbb{C}_2 f'(\omega_1) + ...$ throughout all this.

BROWNIAN MOTION

has both the state space, i.e. $X(t)$, and time run on a continuous scale. It is based on the assumptions of continuity and of independent increments, implying that

$$X(t) \in \mathcal{N}(x_0 + d \cdot t, \sqrt{c \cdot t})$$

where d is the drift parameter, and c is the diffusion coefficient.

When $d = 0$, we have:

Probability that a is never reached by time T, or

$$\Pr\left(\max_{0 \le t \le T} X(t) < a \mid X(0) = x_0\right) = 1 - 2\Pr\left(Z > \frac{a - x_0}{\sqrt{c \cdot T}}\right)$$

when $a > x_0$ (with the obvious changes for $a < x_0$).

Probability of $X(T) > y$ without ever reaching z, or

$$\Pr\left(X(T) > y \cap \min_{0 \le t \le T} X(t) > z \mid X(0) = x \right) =$$

$$\Pr\{Z > \frac{y-x}{\sqrt{c \cdot T}}\} - \Pr\{Z > \frac{y+x-2z}{\sqrt{c \cdot T}}\}$$

where $y > z$ and $x > z$ (with the obvious changes for $y < z$ and $x < z$).

Probability of returning to the starting value (at least once) during a $(t_0,\ t_1)$ interval, or

$$\Pr\left(\max_{t_0 < t < t_1} X(t) > x \cap \min_{t_1 < t < t_1} X(t) < x \mid X(0) = x \right) =$$

$$\frac{2}{\pi} \arccos \sqrt{\frac{t_0}{t_1}}$$

When d has any value, we have only one result:

The distribution of $X(t)$, given that ... $X(t_0) = x_0$, $X(t_1) = x_1$, $X(t_2) = x_2$, $X(t_3) = x_3$, ...(where ... $t_0 < t_1 < t < t_2 < t_3 < ....$) is *normal*, with the mean of

$$\frac{x_1(t_2 - t) + x_2(t - t_1)}{t_2 - t_1}$$

(linear interpolation) and the standard deviation of

$$\sqrt{c\,\frac{(t_2 - t)(t - t_1)}{t_2 - t_1}}$$