Questions with the ■ mark should be answered without Maple.

1. Consider a process with rates

$$\begin{aligned} \lambda_n &= 27 + 3\,n \quad \text{per min.} \\ \mu_n &= 5\,n \quad \text{per min.} \end{aligned}$$

   If the process starts with 4 initial members, find:

   (a) ■ The mean and standard deviation of the value of the process 17 second later.

   (b) The probability that, 17 seconds later, the process has between 3 and 7 members (inclusive).

   (c) The long run proportion of time spent in State 10. How long, on the average, are durations of visits to State 10?

   (d) On the average, how often does the process enter State 10 in a long run?

2. This is a continuation of the previous question.

   (a) ■ What is the probability that, at $t = 17$ seconds, at least one of the initial members is still alive?

   (b) ■ Suppose T is the time of death of the last one of the 4 initial members. Compute $\mathbb{E}(T)$ and the corresponding standard deviation.

   (c) What is the probability that, during the first 17 seconds, more than 10 immigrants arrive?

   (d) What is the probability that, at $t = 17$ seconds, there is more than 5 immigrants and their descendents still alive?

   (e) What is the probability that, at $t = 17$ seconds, there is more than 5 immigrants (but *not* their descendents) still alive?

3. Consider a $M/M/\infty$ queue where customers arrive at the rate of 18.4 per hour, and a single service takes, on the average, 21 minutes. At 9:00, there are 8 people being serviced.

   (a) What is the probability that by $t =$ 9:32, at least 5 of *these* 8 people have already left?

   (b) What is the probability that at $t =$ 9:32 there will be at least 9 people in service. Also, compute $\mathbb{E}\left[X(9{:}32)\right]$ and the corresponding standard deviation.

   (c) ■ How often does it happen, in a long run, that all servers become idle (give the average number of such occurrences per 7 day week, assuming a 24/7 operation).

4. Consider a Birth-and-Death process with the following rates

$$\begin{aligned} \lambda_n &= 27 - 3\,n \quad \text{per hour} \\ \mu_n &= 5\,n \qquad \text{per hour} \end{aligned}$$

where $n = 0,\ 1,\ 2,\ ..., 9$.

(a) If the process starts in State 4, what is the probability that 8 minutes later the process is in State 5?

(b) ■ Find the long-run proportion of time the process spends in State 5.

(c) What is the expected time between two consecutive visits to State 5 (entry to entry)?

5. ■ Solve

$$\dot{P}(z,t)\cos^2 z = P'(z,t)\cos z + P(z,t)\sin z$$

subject to the following initial condition:

$$P(z,0) = \cos z \cdot \sin^2 z$$