BROCK UNIVERSITY

Final Examination: April 2009	Number of Pages: 3
Course: MATH 4F85	Number of students: 9
Date of Examination: Apr. 16, 2009	Number of Hours: 3
Time of Examination: 9:00-12:00	Instructor: J. Vrbik

Full credit given for **7** correct and complete answers.
One sheet of notes and a Maple workspace (loaded from a memory stick) are allowed.

1. Customers arrive at a 'looney' store (every item costs \$1) at an average rate of 19.2 per hour. Each of them spends a random (from our point of view) amount of money which has the *modified* geometric distribution with $p = 0.32$. Find

   (a) the probability that more than 5 'browsers' (customers who make no purchase) arrive during the next forty minutes,

   (b) the expected value and standard deviation of the total amount of money spent by customers who arrive between 8:25 and 9:02,

   (c) the probability that this amount (from part b) is bigger than \$40.

2. For a continuous-time Markov process with the following (per hour) rates

$$\begin{bmatrix} \times & 2.1 & 3.5 & 1.0 \\ 0.9 & \times & 2.3 & 1.8 \\ 3.3 & 1.1 & \times & 2.0 \\ 4.1 & 0.7 & 2.7 & \times \end{bmatrix}$$

   (a) find the corresponding (*exact*, i.e. using fractions) stationary distribution

   (b) and the average (long-run) frequency of visits to the first state per day (24 hours).
   The process is currently in the second state. Find

   (c) the expected time till the first visit (from now, until entry) to the last (fourth) state, and the corresponding standard deviation,

3. Consider a Brownian motion with no drift, and the diffusion coefficient of 7.9 $\frac{\text{mm}^2}{\text{hr}}$. If $X(8{:}25) = 5$ mm, compute the probability that the process

   (a) will have a positive value at 12:08,

   (b) will have had negative values at least once, during the 8:25 to 12:08 interval (regardless of the final value),

   (c) will have a negative value at 12:08, without ever reaching 9 mm.

4. Consider an $M/M/5$ queue with an average service time of 9 min. and 21 sec., and customers arriving at an average rate of 27.3 per hour. Compute the long-run

   (a) server utilization factor,

   (b) average waiting time,

   (c) how often does it happen (on the average, per day - assume a non-stop operation) that all servers are idling.

5. Solve

$$\ln(z)\dot{P}(z,t) = z^2 P'(z,t) + zP(z,t)$$

   subject to

$$P(z,0) = \frac{1}{1+\ln(z)}$$

6. Consider Brownian motion with drift and diffusion coefficients (d and c) equal to 14.2 $\frac{\text{m}}{\text{day}}$ and 127$\frac{\text{m}^2}{\text{day}}$ respectively. Compute

   (a) $\Pr[X(20{:}47) > 17.3\text{m} \mid X(8{:}12){=}2.2\text{m} \cap X(11{:}39){=}6.4\text{m} \cap X(15{:}04){=}11.4\text{m}]$,

   (b) $\Pr[X(11{:}39) > 6.4\text{m} \mid X(8{:}12){=}2.2\text{m} \cap X(15{:}04){=}11.4\text{m} \cap X(20{:}47){=}17.3\text{m}]$,

   (c) $\Pr[X(15.04) > 11.4\text{m} \mid X(8{:}12){=}2.2\text{m} \cap X(11{:}39){=}6.4\text{m} \cap X(20{:}47){=}17.3\text{m}]$.

7. A continuous-time Markov process with 5 states (labelled $\underline{1}$ to $\underline{5}$; the first and last are *absorbing*), has the following transition rates (per hour):

To → / From ↓	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{2}$	0.9	×	2.3	1.8	1.0
$\underline{3}$	3.3	1.1	×	2.0	0.4
$\underline{4}$	4.1	0.8	2.7	×	0.9

   If the process starts in State $\underline{3}$, find

   (a) the probability that, 14 minutes later, the process is in State $\underline{4}$,

   (b) the *exact* (i.e. fraction) probability of being absorbed (sooner or later) by State $\underline{1}$,

   (c) expected time till absorption (in either absorbing state) and the corresponding standard deviation.

8. Consider a process consisting of 'bacteria' which *individually* procreate (i.e. split in two) at the rate of 0.57 per day, and have an average life span of 1 days and 12 hours (exponentially distributed). We start with a colony of 9 individuals, who are replenished by random 'immigration' at an average rate of 2.7 per day. Compute

   (a) the probability that, two days later, the process consists of more than 13 bacteria,

   (b) the long-run average number of living bacteria,

   (c) the expected time till extinction of the 'native' sub-population.

9. Consider a Birth and Death process with the following (per minute) rates

$$\begin{aligned}\lambda_n &= \frac{4.2\ n}{1+n} \\ \mu_n &= 2.8\ \ln(1+n)\end{aligned}$$

   Given that the process is in State 10 at 8:21:46, find the probability that it will get (sooner or later) trapped in State 0 (note that State 0 is absorbing). If you discover (by proper procedure - show the details) that this probability is equal to 1, find the expected time of absorption (express it in the xx:yy:zz format).

10. Consider an $M/G/\infty$ queue with customers arriving at an average rate of 9.7 per hour, and the service time having a distribution with the following probability density function

$$f(t) = \begin{cases} \frac{t-20}{400}\exp(1-\frac{t}{20}) & 20 < t \\ 0 & \text{otherwise} \end{cases}$$

   where t is time in *minutes*. At time zero, there are no customers in the system. Compute

   (a) the probability that, 55 minutes later, more than 4 people are being serviced, while at least 3 have already left,

   (b) the long-run average number of customers in the system,

   (c) the long-run average frequency of visits to State 0 (no customers), per year (365 days).

11. Solve

$$\tan(z)\dot{P}(z,t) = \cos(z)P\,'(z,t)$$

   subject to

$$P(z,t) = z$$

12. Find arctan of the following matrix

$$\begin{bmatrix} -1 & -5 & 5 & -5 \\ 5 & 9 & -5 & 5 \\ 5 & 5 & -1 & 5 \\ -5 & -5 & 5 & -1 \end{bmatrix}$$

   (use *decimal* numbers for your answer).