BROCK UNIVERSITY

Final Examination: April 2013	Number of Pages: 4
Course: MATH 4F85	Number of students: 6
Date of Examination: Apr. 15, 2013	Number of Hours: 3
Time of Examination: 9:00-12:00	Instructor: J. Vrbik

Open book exam. Use of Maple is allowed.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Translation dictionaries (e.g. English-French) or other dictionaries (thesaurus, definitions, technical) are not allowed unless specified by the instructor and indicated on the examination paper.

Full credit given for 22 (out of 33) complete answers.

Numerical answers must be correct to 4 significant digits.

---

1. Consider an $M/G/\infty$ queue with customers arriving at an average rate of 7.2 per hour, and the service time of each customer taking exactly 12 minutes. At time zero, there are no customers in the system. Compute

   (a) the probability that, 52 minutes later, fewer than 4 people are being serviced, while more than 4 have already left,

   (b) the long-run average of busy servers.

   (c) In the long run, how often (on the average, per hour) does the system enter State 0 (with no customers)?

2. Consider a B&D process with the following (per hour) transition rates

n	0	1	2	3	4	5	6
λ_n	1.3	2.0	1.4	2.1	2.5	1.3	0
μ_n	0	2.3	4.2	2.2	1.3	1.9	2.0

   Find:

   (a) the corresponding (exact, i.e. using fractions) stationary distribution,

   (b) the long-run frequency of visits to State 3 (the forth state of our table),

   (c) If the process is currently in State 3, what is the probability that, 12 minutes later, the process will be in State 5, without ever (during those 12 minutes) visiting State 1?

3. Consider a LGWI process with the initial value 6. The expected lifetime of each 'member' of this process is 23 days, each living member splits in two on at the average rate of once every 31 days, and the rate of immigration is 0.12 members per day.

   (a) What is the probability of having fewer than 5 surviving immigrants (not counting their descencents) 52 days later?

   (b) What is the probability of having fewer than 5 surviving immigrants (including their descencents) 52 days later?

   (c) Find the expected time till extinction of the 'native' sub-population (the initial 6 members and their progeny), and the corresponding standard deviation.

4. Consider a B&D process with the following (per minute) rates

$$\lambda_n = 0.6\sqrt{n}$$
$$\mu_n = \frac{3n}{1+n}$$

   Given that the process is now in State 16, find

   (a) the probability the process becomes extinct,

   (b) the probability that the process never reaches State 10,

   (c) the probability that the process never reaches State 20.

5. Customers arrive at a store at an average rate of 3.8 'clusters' per hour. The cluster sizes are independent of each other, and have a distribution with the following probability function

$$f(i) = \frac{4.3^{i-1}}{(i-1)!}e^{-4.3} \qquad \text{where } i \text{ is a non-negative integer}$$

   The time is 9:27. Find

   (a) the probability that the 3^{rd} cluster (from now) will arrive before 9:52,

   (b) the probability that more than 13 customers will arrive at the store between now and 9:52,

   (c) the probability that the next two clusters (whenever they happen) will bring in a total of at least 5 customers.

6. Assuming
$$\mathbb{A} = \begin{bmatrix} -1 & -1 & -1 & 2 \\ 3 & 4 & 1 & -1 \\ 0 & -2 & 2 & -2 \\ -4 & -1 & -1 & 5 \end{bmatrix}$$
(note the matrix has only simple integer eigenvalues) compute:
   (a) $\ln(\mathbb{A})$,
   (b) $\sin(\mathbb{A})$.
   (c) Verify *both* answers using the exponential function of Maple.

7. Consider an $M/M/5$ queue with the arrival rate of 14.2 customers per hour, and the mean service time of 18 minutes. Compute the long-run:
   (a) server utilization factor,
   (b) average waiting time (in minutes and seconds).
   (c) Assuming the process is now in State 5 (all servers busy), what is the expected time till entering State 4 (one server idling)?

8. Find, without Maple, the general solution to
   (a)
$$ze^z\dot{P}(z,t) = P'(z,t)$$
   (b)
$$ze^z\dot{P}(z,t) - P'(z,t) = 3P(z,t)$$
   (c) and the particular solution to the last equation (Part b) which meets the following initial condition:
$$P(z,0) = (z-1)^3$$

9. Consider an $M/M/\infty$ queue with cutomers arriving at an average rate of 26.3 per hour, the mean service time of 12 minutes and 37 seconds, and the initial value of 8 customers. Find the expected time till the process enters (for the first time from now)
   (a) State 5 (with 5 customers),
   (b) State 12,
   (c) State 8 again (unlike Parts a and b, there is a simple formula to answer *this* question).

10. Consider a 3D Poisson process with the average density of 'points' equal to 0.098 per meter^3. Find

   (a) the expected distance to the forth nearest point from the origin, and the corresponding standard deviation,

   (b) the expected number of points inside the region defined by

$$x^2 - 2x + y^2 + 4y < 5 \qquad \text{and} \qquad -2 < z < 3$$

   and the corresponding standard deviation (meters are the units of each axis).

   (c) Given that there are exactly 18 points inside the region of Part *b*, what is the conditional probability that fewer than 8 of them have a negative z coordinate?

11. Consider a continuous-time Markov chain with the following (per hour) rates

$$\begin{bmatrix} \times & 2 & 2 & 3 & 1 \\ 4 & \times & 2 & 0 & 2 \\ 3 & 4 & \times & 6 & 1 \\ 4 & 0 & 3 & \times & 5 \\ 2 & 2 & 1 & 5 & \times \end{bmatrix}$$

(let us label the states 1, 2, ... 5). Assuming that now the process is in State 2, find the *exact* (not decimal) value of

   (a) the probability of visiting State 4 *before* vising State 1,

   (b) the expected time and the corresponding standard deviation of the time of the first (from now) entry to State 5,

   (c) the probability of returning to State 2 before visiting State 1.