BROCK UNIVERSITY

Final Examination: April 2015 | Number of Pages: 4
Course: MATH 4F85 | Number of students: 4
Date of Examination: Apr. 15, 2013 | Number of Hours: 3
Time of Examination: 10:00 - 13:00 | Instructor: J. Vrbik

Use of Maple is allowed.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Translation dictionaries (e.g. English-French) or other dictionaries (thesaurus, definitions, technical) are not allowed unless specified by the instructor and indicated on the examination paper.

Full credit given for 22 (out of 33) complete answers.
Numerical answers must be correct to at least 4 significant digits.
Proper units must be given whenever appropriate.
All answers must be entered in the exam booklet.
Maple work should be printed and attached.

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1. Consider a 3D Poisson process with the average number of 'dots' equal to 17.3 per cubic meter (let meter be our unit of length). Find

   (a) the expected number of dots found in the following region

   $$x^2 + y^2 + z^2 - 2x + 4y + 12z < 40$$

   and the corresponding standard deviation.

   (b) the expected distance from the origin to its fifth nearest dot, and the corresponding standard deviation.

   (c) the probability that the region defined by

   $$x^2 + y^2 < 1 \qquad \text{and} \qquad 0 < z < 3$$

   contains the same number of dots as the region

   $$x^2 + y^2 < 1 \qquad \text{and} \qquad -3 < z < 0$$

2. Consider the $M/G/\infty$ queue with the average arrival rate of 45.7 customers per hour and service times having the uniform distribution between 5 and 13 minutes (each). Find

   (a) the expected number of new customers arriving during the service time of the 15^{th} customer, and the corresponding standard deviation,

   (b) the long-run proportion of time with more than 10 busy servers,

   (c) the expected number of busy servers 10 minutes after opening (with no customers waiting), and the corresponding standard deviation.

3. Consider the LGWI model with $\lambda = \mu = 0.027$ per year (these are the *individual* birth and death rates) and the average immigration rate of 0.36 per year. Assuming that the process starts with 184 individuals, find

   (a) the expected time of death (and the corresponding standard deviation) of the last one of the 184 initial members,

   (b) the probability that, 1500 years later, there are no 'natives' (the initial members and their progeny) left,

   (c) the expected value and standard deviation of the number of immigrants and their descendents who are alive 1500 years later.

4. Consider the $M/M/10$ queue with customers arriving at an average rate of 37.8 per hour and the average service time of 13 minutes and 28 seconds. Compute the long-run

   (a) proportion of time with all servers busy,

   (b) average waiting time (in minutes).

   (c) Starting in State 0, find the expected time and the corresponding standard deviation (both in minutes) of the first entry to State 10 (all servers busy). Hint: make State 10 absorbing.

5. Customers arrive at a store in groups (called clusters), at an average rate of 8.2 *clusters* per hour. The size of each cluster is a random variable with the following distribution:

Number of customers:	1	2	3	4
Pr:	0.40	0.35	0.20	0.05

   Find the probability

   (a) of fewer than 12 *clusters* arriving during the next 57 minutes,

   (b) of fewer than 12 *customers* arriving during the next 57 minutes,

   (c) that *none* of the the clusters arriving during the next 57 minutes will be of size 4.

6. Consider an $M/M/1$ queue with the average arrival rate of 12.8 customers per hour, the average service time of 13 minutes and 28 seconds, and the probability that an arriving customer joins the queue is $\frac{1}{1+k}$, where k is the number of customers *waiting* for service (at the time of the new arrival). Compute the long-run

   (a) server utilization factor,

   (b) percentage of lost customers,

   (c) frequency of visits to State 0 (per 24-hour day).

7. For the following PDE

$$(1+z)\cdot \dot{P}(z,t) = e^z \cdot \big(P'(z,t) - P(z,t)\big)$$

   find

   (a) the general solution,

   (b) solution which meets $P(z,0) = z+2$.

   (c) Given the following matrix

$$\mathbb{A} = \begin{bmatrix} 2 & -1 & -1 \\ 2 & 2 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

   find a formula for computing $\mathbb{A}^n$ (where n is any real number).

8. Consider the B&D process with 14 welders (each welder using electricity for the average duration of 16 seconds and being idle for 27 seconds). Given that the the system is currently in State 6, find the probability of

   (a) being in State 8 half a minute later,

   (b) visiting State 8 before visiting State 4.

   (c) Compute the long-run frequency of visits to State 0 (per hour), and convert it into the expected duration of a busy period (defined as the time interval during which at least one welder is using current).

9. Consider a Poisson process with the (per hour) arrival rate at time t (in hours, taking 8:00 as $t = 0$) given by

$$\lambda(t) = \frac{12.7}{1+t}$$

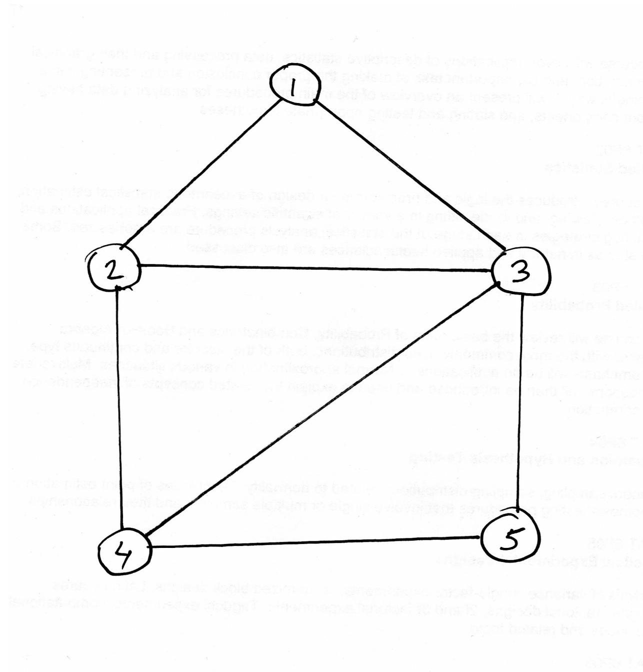
   Find

   (a) the probability of more than 8 arrivals between 9:27 and 10:42,

   (b) the expected time (in the xx:yy:zz format) of the third arrival (since 8:00), and the corresponding standard deviation (in minutes and seconds),

   (c) the probability that the third arrival happens before 8:16.

10. A particle moves from node to node following these two rules: the time to make a move is always exponentially distributed with the mean of 24 seconds (independent of the past), and a move takes the particle to one of its *adjacent* nodes, chosen with equal probability for each connection (indicated by a line). The particle starts in Node 1.



Find

(a) the probability that one minute later the particle is in Node 5,

(b) the expected time to reach Node 5 for the first time, and the corresponding standard deviation,

(c) the probability that the particle visits Node 5 before visiting Node 3.

11. Consider a B&D process with the following per-hour rates

$$\begin{aligned}\lambda_0 &= 0\\ \lambda_n &= 1 \qquad \text{when} \qquad n \geq 1\\ \mu_n &= \frac{99n}{100n+1}\end{aligned}$$

and the initial value of 15. Find the probability of

(a) its ultimate extinction,

(b) never visiting State 10 (hint: make it an absorbing state),

(c) never visiting State 20 (hint: make it an absorbing state).