BROCK UNIVERSITY

Final Examination: December 2019	Number of Pages: 3
Course: MATH 4F85	Number of students: 6
Date of Examination: Dec. 13, 2019	Number of Hours: 3
Time of Examination: 14:00 - 17:00	Instructor: J. Vrbik

Open book exam. Use of Maple is allowed.

No examination aids other than those specified on the examination scripts are permitted (this regulation does not preclude special arrangements being made for students with disabilities). Translation dictionaries (e.g. English-French) or other dictionaries (thesaurus, definitions, technical) are not allowed unless specified by the instructor and indicated on the examination paper.

Full credit given for 18 (out of 31) complete answers.

Numerical answers must be correct to at least 4 significant digits.
Proper units must be given whenever appropriate.
All answers must be entered in the exam booklet.
Maple work should be e-mailed to jvrbik@brocku.ca.

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1. Consider a 3D Poisson process with the average number of 'dots' equal to 15.3 per cubic meter (meter is our unit of length). Find

   (a) the expected number of dots to be found in the following region

   $$x^2 + y^2 + z^2 + 2x - 4y + 11z < 40$$

   and the corresponding standard deviation.

   (b) the expected distance from the origin to its fifth nearest dot, and the corresponding standard deviation.

2. Consider the $M/G/\infty$ queue with the average arrival rate of 57.2 customers per hour, and service times having the *uniform* distribution from 4 to 11 minutes. Find

   (a) the expected number of new customers to arrive during the service time of the 15^{th} customer (from now), and the corresponding standard deviation (hint: use Poisson process of random duration),

   (b) the long-run proportion of time with more than 10 busy servers.

3. Consider the LGWI model with $\lambda = \mu = 0.027$ per year (these are the *individual* birth and death rates) and the average immigration rate of 0.36 per year. Assuming that the process starts with 14 members, find

   (a) the expected time of death (since the start of the process) and the corresponding standard deviation of the *first* one of the 14 initial members to die,

   (b) the probability that, 230 years later, there are no 'natives' (initial members and their progeny) left,

   (c) the expected number and the corresponding standard deviation of members of the immigrant subpopulation (i.e. including all their descendents) 230 years later.

4. Consider the $M/M/8$ queue with customers arriving at an average rate of 27.3 per hour, and the average service time of 12 minutes and 15 seconds. Compute the long-run

   (a) proportion of time with all servers busy,

   (b) average waiting time (in minutes).

5. Customers arrive at a store in groups (called clusters) at an average rate of 7.3 *clusters* per hour. The size of each cluster is a random variable with the following distribution:

Number of customers:	1	2	3	4	5
Pr:	0.37	0.34	0.23	0.05	0.01

   Find the probability

   (a) of fewer than 12 *clusters* arriving during the next 72 minutes,

   (b) of fewer than 12 *customers* arriving during the next 72 minutes,

   (c) that *none* of the the clusters arriving during the next 72 minutes will consist of more than 3 customers.

6. Consider an $M/M/1$ queue with the average arrival rate of 11.3 customers per hour, the average service time of 10 minutes and 54 seconds, and the probability of an arriving customer joining the queue given by $\frac{1}{1+k}$, where k is the number of customers *waiting* for service at the time of his/her arrival. Compute the long-run

   (a) server utilization factor,

   (b) percentage of lost customers,

   (c) average duration of busy *cycles*.

7. For the following PDE

$$z \cdot \dot{P}(z,t) = e^{-z} \cdot \big(P'(z,t) + P(z,t)\big)$$

   find (*without* Maple)

   (a) the general solution,

   (b) the particular solution which meets $P(z,0) = e^z \cdot (1-z)^2$; you must spell out the corresponding $g(x)$.

8. Find a formula (leave it in its constituent-matrix form) for computing $\mathbb{A}^q$, where q is a *real* exponent and

   (a)

$$\mathbb{A} = \begin{bmatrix} -2 & -14 & 13 \\ 11 & 36 & -31 \\ 8 & 29 & -25 \end{bmatrix}$$

   (b)

$$\mathbb{A} = \begin{bmatrix} 2 & 15 & -7 & 27 \\ -1 & 21 & -7 & 35 \\ 0 & 9 & -1 & 18 \\ -1 & -5 & 3 & -8 \end{bmatrix}$$

9. Consider the power-supply process with 12 welders (each welder uses electricity for the average duration of 12 seconds and then stays idle for the average of 19 seconds, indefinitely repeating this cycle). Given that the the system is currently in State 4 (four welders using electricity) find

   (a) the probability of being in State 6 half a minute later,

   (b) the expected time (and the corresponding standard deviation) of the time of the first entry (from now) to State 6.

10. Consider a Poisson process with the (per hour) expected arrival rate at the time t (in hours, taking 8:00 as $t = 0$) given by

$$\lambda(t) = \frac{9.7}{1+t}$$

   Find

   (a) the probability of getting more than 7 arrivals between 9:25 and 10:37,

   (b) the expected time of the fourth arrival (since 8:00), and the corresponding standard deviation.

11. Consider a B&D process with the following (per minute) rates

n	1	2	3	4	5
λ_n	1.3	2.4	1.8	2.4	0
μ_n	0	3.2	2.0	1.7	2.9

   and starting in State 2. Find (in *exact* fractions)

   (a) the long-run proportion of time spent in State 5,

   (b) the expected time and the corresponding *variance* of reaching (for the first time from the start of the process) State 5,

   (c) the probability of reaching State 5 before reaching State 1.

12. Consider a B&D process with the following per-hour rates

$$\lambda_n = \begin{cases} 0 & \text{when} \quad n = 0 \\ 1 & \text{when} \quad n \geq 1 \end{cases}$$

$$\mu_n = \frac{9n^2}{10n^2+1}$$

   and the initial value of 5. Find the probability of

   (a) its ultimate extinction,

   (b) never visiting State 10.

13. Consider a Brownian motion with zero drift and the diffusion coefficient of 13 $\frac{\text{mm}^2}{\text{hr}}$. If the process was observed to have the value of 3.1 mm at 8:13, the value of -1.3 mm at 10:37, and has currently (at 11:09) the value of -1.7 mm, compute the probability that

   (a) it had a positive value at 9:23,

   (b) it will have reached a positive value (at least once) between now and 12:03,

   (c) it will end up with a positive value at 12:03, without ever dipping (between now and then) to a value lower than -2.5 mm.