BROCK UNIVERSITY

Final Examination: April 2004	Number of Pages: 3
Course: MATH 4F85	Number of students: 8
Date of Examination: Apr. 16, 2004	Number of Hours: 3
Time of Examination: 9:00-12:00	Instructor: J. Vrbik

This is an open-book exam. Full credit given for **7** correct and complete answers.

1. Consider an $M/G/\infty$ queue where service times are of random duration, having the gamma(2, 12 min.) distribution, and customers are arriving at the rate of 13.2 per hour. Find:

   (a) probability that more than 5 customers will be serviced half an hour after opening (assuming we started with no customers)

   (b) probability that, one hour after opening, more than 5 customers have completed their service,

   (c) the long-run proportion of time with more than 5 customers being serviced,

2. Consider an $M/M/\infty$ queue with customers arriving at the rate of 13.2 per hour and the average service time of 24 minutes. Also assume that, at the opening time, there are three customers already waiting (this is the initial state). Compute:

   (a) the expected value and the corresponding standard deviation of the number of customers being serviced half an hour after opening,

   (b) probability that more than 5 customers will be serviced half an hour after opening,

   (c) the long-run proportion of time with more than 5 customers being serviced.

3. For a continuous-time Markov process with the following (per hour) rates

$$\begin{bmatrix} \times & 1.4 & 2.6 & 0.4 \\ 3.2 & \times & 1.9 & 0.8 \\ 2.0 & 3.5 & \times & 1.2 \\ 1.1 & 0.5 & 2.6 & \times \end{bmatrix}$$

   (a) find the corresponding stationary distribution.

   (b) If the process is currently in State 2, what is the probability that the process will be, 25 minutes later, in State 3?

   (c) How often is State 3 visited, on the average, every day (24 hours)?

4. Consider a Birth and Death process with the following rates (per minute)

State:	0	1	2	3	4	5	6	7
λ_n	0	3.3	4.1	0.9	1.3	2.5	1.9	0
μ_n	0	2.3	3.0	4.1	0.8	1.2	2.3	0.2

   (note that State 0 is absorbing). Given that the process is now in State 6, find the expected time till absorption. What is the expected duration of a visit to State 3 (i.e. from entring it till leaving it)?

5. Consider a Brownian motion with a drift of $-5.2\frac{\text{mm}}{\text{hr}}$ and a diffusion coefficient of $7.3\frac{\text{mm}^2}{\text{hr}}$. Evaluate:

   (a) $\Pr\{X(10{:}23) < 26\,\text{mm}\,|\,X(9{:}08) = 30\,\text{mm} \cap X(8{:}42) = 32\,\text{mm}\}$

   (b) $\Pr\{X(10{:}23) < 26\,\text{mm}\,|\,X(9{:}08) = 30\,\text{mm} \cap X(10{:}42) = 23\,\text{mm}\}$

6. Find the general formula (in its real form) for computing all serial correlation coefficients of the following model

$$X_i = 1.2\,X_{i-1} - 0.7\,X_{i-2} + \epsilon_i$$

   where $\epsilon_i \in \mathcal{N}(0, 1.35)$. Also, compute the stationary variance of X_i, and

$$\Pr(X_{57} > 3 | X_{56} = 2.3 \cap X_{55} = -0.4 \cap X_{54} = 1.0 \cap X_{53} = 1.8)$$

   Why is this model stable?

7. Consider a Birth and Death process with $\lambda_n = 7.4 \times n$ per hour, $\mu_n = 7.7 \times n$ per hour, and $X(0) = 4$. Compute:

   (a) expected value and standard deviation of $X(15 \text{ min.})$,

   (b) probability that $X(15 \text{ min.}) > 7$,

   (c) expected time till extinction,

   (d) probability that extinction will take more than 4 hours.

8. Solve

$$\dot{P}(z,t)\cos^2 z = P\,'(z,t)\cos z + P(z,t)\sin z$$

   subject to the following initial condition:

$$P(z,0) = \frac{\cos z}{\sin z}$$

9. Consider a Brownian motion with no drift and the diffusion coefficient of $31.7\frac{\text{m}^2}{\text{hour}}$. Starting at 8:00. with the value of 13.9 m, what is the probability that the process will

   (a) have a negative value at 11:00.

   (b) avoid zero until 11:00.

   (c) avoid zero until 11:00, at which time it will reach a value smaller than 10.4 m.

   (d) return to 13.9 m, at least once, between 11:00 and 12:15.

10. Find the eigenvalues and the corresponding constituent matrices of

$$\mathbb{P} = \begin{bmatrix} -8 & 3 & 0 & -6 \\ 36 & -15 & -10 & 36 \\ 12 & -6 & -2 & 12 \\ 27 & -11 & -5 & 25 \end{bmatrix}$$

Using these, evaluate $\cos(\mathbb{P})$.

11. Find the first 8 (up to and including ρ_8) serial correlation coefficients of the following autoregressive model

$$X_i = 2.7\,X_{i-1} - 2.59\,X_{i-2} + 0.873\,X_{i-3} + \epsilon_i$$

where $\epsilon_i \in \mathcal{N}(0, 1.79)$. Also, compute the stationary variance of X_i, and

$$\Pr(X_{107} > 2.14 \mid X_{105} = -0.31)$$

Is this process stable and why?