BROCK UNIVERSITY

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Course: MATH 4F85	Number of students: 7
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Time of Examination: 19:00-21:00	Instructor: J. Vrbik

This is an open-book exam. Full credit given for **7** correct and complete answers.

1. Customers arrive at a 'looney' store (every item costs \$1) at an average rate of 29.2 per hour. Each of them spends a random (from our point of view) amount of money having the binomial distribution with parameters $n = 6$ and $p = 0.32$. Find
   (a) the probability that more than 5 customers will arrive at the store during the next ten minutes,
   (b) the expected value and standard deviation of the total amount of money spent by customers who arrive between 8:15 and 8:36,
   (c) the probability that this amount (from part b) is bigger than \$30.

2. Consider a pure-death process which starts with 18 individuals. They each live (individually, and independently of one another) for a time which is exponentially distributed, with the mean of 5 hours and 17 minutes. Compute:
   (a) the expected number and the corresponding standard deviation of surviving individuals, 7 hours later,
   (b) the probability that this number (of 7 hour survivors) is bigger than 5,
   (c) the expected value and standard deviation of the time till the process' extinction.

3. For a continuous-time Markov process with the following (per hour) rates (assume that the states are labelled 0 to 3)

$$\begin{bmatrix} \times & 2.1 & 3.5 & 1.0 \\ 0.9 & \times & 2.3 & 1.8 \\ 3.3 & 1.1 & \times & 2.0 \\ 4.1 & 0 & 2.7 & \times \end{bmatrix}$$

   (a) find the corresponding (exact, i.e. using fractions) stationary distribution.
   (b) If the process is currently in State 0, what is the probability that, 17 minutes later, the process will be in State 3?
   (c) In the long run, how often is State 2 visited, on the average, every hour?

4. Consider a Birth and Death process with the following (per minute) rates

State:	0	1	2	3	4	5	6	7
λ_n	2.9	3.3	4.1	0.9	1.3	2.5	1.9	0
μ_n	0	2.3	3.0	4.1	0.8	1.2	2.3	0.8

   Find the corresponding stationary distribution. In the long run, what is the average value of the process? How often is State 2 visited, on the average, every hour?

5. Consider a Brownian motion with no drift, and a diffusion coefficient of 7.3 $\frac{\text{mm}^2}{\text{hr}}$. If the process is initially at -3 mm, compute the probability that, 2.5 hours later, the process

   (a) will have a positive value,

   (b) will have had a positive value (i.e. has visited 0), at least once, during the 2.5 hours, regardless of what the final value is,

   (c) will have a positive value (at the end of the 2.5 hours), without ever dipping below -4 mm.

6. Consider the following autoregressive model:

$$X_n = 1.4\,X_{n-1} - 0.8\,X_{n-2} + \epsilon_i$$

   where ϵ_n are independent, normally distributed, random variables with the mean of 0, and standard deviation of 2.1 .

   (a) Is this process stable (substantiate your answer)?

   (b) Compute, numerically, its first 10 serial correlation coefficients (up to and including ρ_{10}).

   (c) What is the stationary distribution of X_n? Based on this, compute $\Pr(X_{517} > 5)$.

7. Consider an $M/M/6$ queue with the average service time of 7 min. and 22 sec., and customers arriving at an average rate of 31.4 per hour. Compute (the long-run)

   (a) server utilization factor,

   (b) average size of the line up,

   (c) percentage of time with no line up,

   (d) percentage of time all servers are busy.

8. Solve

$$2z\,\dot{P}(z,t) = (z^2 - 1)\;P'(z,t) + P(z,t)$$

   subject to the following initial condition:

$$P(z,0) = 1 + z$$

9. Consider a birth and death process with $\lambda_n = 3.2\;n^2$ and $\mu_n = 3.4\;n^2$ (both per hour), and the initial value of 13.

   (a) Is extinction of this process certain, or is there some chance of ultimate survival?

   (b) If extinction is certain, compute the expected duration of time to reach it (and ignore Part c).

   (c) Otherwise, compute the probability of the process' ultimate survival.

10. Consider a process which consists of 'bacteria' which are unable to procreate, and have (individually) an average life span of 2 days and 3 hours (exponentially distributed). We start with a colony of 5 individuals, who are replenished by random 'immigration' at an average rate of 1.8 per day.

    (a) This is clearly an example of a birth-and-death process. What are its λ_n and μ_n rates (and how did we call the process with such rates)?

    (b) What is the probability that, two days later, the process consists of fewer than 4 bacteria?

    (c) What is the long-run average value of the process, and the corresponding standard deviation?

    (d) How often does the process enter State 0 (find the long-run average, per 365 days).

11. Consider an $M/G/\infty$ queue with customers arriving at an average rate of 16.1 per hour, and the service time having a uniform distribution, spread between 30 and 45 minutes.

    (a) What is the probability that, 2 hours later, more than 5 customers have completed their service?

    (b) What is the probability that, 2 hours later, more than 20 customers are being serviced?

    (c) Find the long-run average number of customers in the system, and the corresponding standard deviation.