BROCK UNIVERSITY

Final Examination: April 2006	Number of Pages: 3
Course: MATH 4F85	Number of students: 7
Date of Examination: Apr. 15, 2006	Number of Hours: 3
Time of Examination: 9:00-12:00	Instructor: J. Vrbik

This is an open-book exam. Full credit given for **7** correct and complete answers.

1. Customers arrive at a store at an average rate of 9.2 'clusters' per hour. The cluster sizes are independent of each other, and have a distribution with the following probability function

$$f(i) = \binom{i+2}{2} \times 0.4^3 \times 0.6^i \qquad \text{where } i \text{ is a non-negative integer}$$

Find

(a) the probability that more than 5 customers arrive at the store during the next seventeen minutes,

(b) the probability that more than 5 clusters arrive at the store during the next seventeen minutes,

(c) the probability that the second last cluster of the day arrives between 4:50 and 4:55 pm. (the store closes at 5).

2. For a continuous-time Markov process with the following (per hour) rates (assume that the states are labelled 0 to 4)

$$\begin{bmatrix} \times & 2.1 & 3.5 & 1.0 & 0.8 \\ 0.9 & \times & 2.3 & 1.8 & 1.0 \\ 3.3 & 1.1 & \times & 2.0 & 0 \\ 4.1 & 0 & 2.7 & \times & 0.9 \\ 2.5 & 1.2 & 0 & 1.5 & \times \end{bmatrix}$$

(a) find the corresponding (exact, i.e. using fractions) stationary distribution.

(b) If the process is currently in State 3, what is the probability that, 13 minutes later, the process will be in State 0?

(c) In the long run, what is the average time between two consecutive visits to State 2 (measured from the time State 2 is left, to the time it is entered again)?

3. Consider a Birth and Death process with the following (per minute) rates

$$\begin{aligned} \lambda_n &= 3.7 \exp(-\frac{n}{5}) \\ \mu_n &= \frac{4.1\, n}{1+n^2} \end{aligned}$$

Is this process stationary? If yes, find the expected value and standard deviation of the stationary distribution. In the long run, how often is State 10 visited, on the average, per hour?

4. Consider a Birth and Death process with the following (per minute) rates

$$\begin{aligned}\lambda_n &= 0.84\ \ln(1+n) \\ \mu_n &= \frac{4.1\ n}{1+n}\end{aligned}$$

Given that the process is now in State 10, find the probability that it will get (sooner or later) trapped in State 0 (note that State 0 is absorbing). If this probability is equal to 1, find the expected length of time till absorption (starting in State 10).

5. Consider a Brownian motion with no drift, and a diffusion coefficient of $5.3\ \frac{\text{mm}^2}{\text{hr}}$. If the process is initially at 4 mm, compute the probability that, 3.5 hours later, the process

   (a) will have a negative value,

   (b) will have had a negative value (i.e. has visited 0), at least once, during the 3.5 hours, regardless of what the final value is,

   (c) will have a negative value (at the end of the 3.5 hours), without ever reaching 6 mm.

6. Find the eigenvalues and the corresponding constituent matrices of

$$\mathbb{P} = \begin{bmatrix} 2 & 1 & \frac{2}{3} \\ \frac{2}{3} & 2 & -\frac{2}{3} \\ 1 & -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$$

Using these, evaluate $\ln(\mathbb{P})$.

7. Consider an $M/M/5$ queue with the average service time of 11 min. and 42 sec., and customers arriving at an average rate of 21.3 per hour. Compute the long-run

   (a) server utilization factor,

   (b) average size of the line up,

   (c) percentage of time with no line up,

   (d) percentage of time all servers are busy.

8. Solve

$$z\ \dot{P}(z,t) = (1+z^2)\ P'(z,t) + z\ P(z,t)$$

subject to the following initial condition:

$$P(z,0) = 1 + z^2$$

9. Consider a birth and death process with $\lambda_n = 48 - 3.2\ n$ and $\mu_n = 3.4\ n$ (both per hour), and the initial value of 8.

   (a) What is the probability that, 25 minutes later, the process is in State 7 ?

   (b) Find the expected value of the process 25 minutes later, and the corresponding standard deviation.

   (c) In the long run, how often (on the average) is State 7 visited per day.

10. Consider a process which consists of 'bacteria' which, individually, procreate (i.e. split in two) at the rate of 0.37 per day, and have an average life span of 1 day and 23 hours (exponentially distributed). We start with a colony of 7 individuals, who are replenished by random 'immigration' at an average rate of 2.1 per day.

    (a) What is the probability that, three days later, the process consists of fewer than 9 bacteria?

    (b) What is the expected number of bacteria three days later, and the corresponding standard deviation?

    (c) How often does the process enter State 0, in the long run, per year (365 days).

11. Consider an $M/G/\infty$ queue with customers arriving at an average rate of 6.3 per hour, and the service time having a distribution with the following probability density function

$$f(x) := \begin{cases} x - 1 & \text{when} \quad 1 < x < 2 \\ 3 - x & \text{when} \quad 2 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

    (x is time in hours). At time zero, there are no customers in the system.

    (a) What is the probability that, 2.5 hours later, more than 6 customers have completed their service?

    (b) What is the probability that, 2.5 hours later, more than 15 customers are being serviced?

    (c) Find the expected number of customers being serviced 2.5 hours later, and the corresponding standard deviation.